

HYPERBOLIC QUASIPERIODIC SOLUTIONS OF U-MONOTONE SYSTEMS ON RIEMANNIAN MANIFOLDS

IGOR PARASYUK

ABSTRACT. We consider a second order non-autonomous system which can be interpreted as the Newtonian equation of motion on a Riemannian manifold under the action of time-quasiperiodic force field. The problem is to find conditions which ensures: (a) the existence of a solution taking values in a given bounded domain of configuration space and possessing a bounded derivative; (b) the hyperbolicity of such a solution; (c) the uniqueness and, as a consequence, the quasiperiodicity of such a solution. Our approach exploits ideas of Ważewski topological principle. The required conditions are formulated in terms of an auxiliary convex function U . We use this function to establish the Landau type inequality for the derivative of solution, as well as to introduce the notion of U -monotonicity for the system. The U -monotonicity property of the system implies the uniqueness and the quasiperiodicity of its bounded solution. We also find the bounds for magnitude of perturbations which do not destroy the quasiperiodic solution.

The results obtained are applied to study the motion of a charged particle on a unite sphere under the action of time-quasiperiodic electric and magnetic fields.

1. INTRODUCTION

Let $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ be a smooth complete connected m -dimensional Riemannian manifold with the metric tensor $\mathbf{g} = \langle \cdot, \cdot \rangle$, and let ∇ be the Levi-Civita connection with respect to \mathbf{g} . For a given smooth mapping $x(\cdot) : I \mapsto \mathcal{M}$ of an interval $I \subset \mathbb{R}$, denote by $\nabla_{\dot{x}} \dot{x}(t)$ the covariant derivative of tangent vector field $\dot{x}(\cdot) : I \mapsto T\mathcal{M}$ along $x(\cdot)$ at the point $t \in I$. Here $T\mathcal{M} = \bigsqcup_{x \in \mathcal{M}} T_x \mathcal{M}$ stands for the total space of the tangent bundle with natural projection $\pi(\cdot) : T\mathcal{M} \mapsto \mathcal{M}$, and $T_x \mathcal{M} = \pi^{-1}(x)$ denotes the tangent space to \mathcal{M} at x .

This paper aims to study a time-quasiperiodic second-order system

$$\nabla_{\dot{x}} \dot{x} = f(t\omega, x), \quad (1.1)$$

as well as its perturbation

$$\nabla_{\dot{x}} \dot{x} = f(t\omega, x) + P(t\omega, x)\dot{x}, \quad (1.2)$$

where $f(\cdot, \cdot) : \mathbb{T}^k \times \mathcal{M} \mapsto T\mathcal{M}$ is a smooth mapping generating the smooth family of vector fields $\{f(\varphi, \cdot)\}_{\varphi \in \mathbb{T}^k}$ on \mathcal{M} parametrized by points of the standard k -dimensional torus $\mathbb{T}^k := \mathbb{R}^k / 2\pi\mathbb{Z}^k$, $\{P(\varphi, \cdot)\}_{\varphi \in \mathbb{T}^k}$ is a smooth family of $(1, 1)$ -tensor fields, and $\omega \in \mathbb{R}^k$ is the basic frequency vector with rationally independent components. Systems of such a kind naturally appear as the Newtonian equations of motion for holonomic mechanical systems undergoing quasiperiodic excitations and perturbations which are linearly dependent upon velocity.

E.g., consider a mechanical system in Euclidean space $\mathbb{E}^N = (\mathbb{R}^N, (\cdot, \cdot))$ endowed with the inner product (\cdot, \cdot) . One can introduce coordinates $y = (y_1, \dots, y_N)$ in such a way that system's kinetic energy be represented as

$$\frac{1}{2}(\dot{y}, \dot{y}) = \frac{1}{2} \sum_{i=1}^N \dot{y}_i^2.$$

2010 *Mathematics Subject Classification.* 37C55; 34C40; 37C65.

Key words and phrases. Newtonian equation of motion; Quasiperiodic solution; Riemannian manifold; Monotone system; Exponential dichotomy .

Suppose that after imposing constraints the system's configuration space turns into an n -dimensional submanifold \mathcal{M} embedded into \mathbb{E}^N by means of inclusion map $\iota : \mathcal{M} \hookrightarrow \mathbb{E}^N$. The inner product (\cdot, \cdot) induces on \mathcal{M} the metric tensor $\mathbf{g} = \langle \cdot, \cdot \rangle := (\iota' \cdot, \iota' \cdot)$, where $\iota' : T\mathcal{M} \mapsto \mathbb{E}^N$ is the derivative of ι , and thus the kinetic energy of constrained system becomes

$$T(\dot{x}) = \frac{1}{2} (\iota'(x)\dot{x}, \iota'(x)\dot{x}) = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle.$$

Let $\Phi(t\omega, y, \dot{y})$ be the resultant force acting on the system and $F(t\omega, x, \dot{x})$ be the generalized force correctly defined by the relation

$$(\Phi(t\omega, \iota(x), \iota'(x)\dot{x}), \iota'(x)\xi) = \langle F(t\omega, x, \dot{x}), \xi \rangle$$

which is required to be true for any vector $\xi \in T_x\mathcal{M}$. It turns out that according to the well known variational principle of analytical mechanics (see, e.g., [23]) the equation of motion in coordinate-independent form can be represented as

$$\nabla_{\dot{x}}\dot{x} = F(t\omega, x, \dot{x}).$$

(In local coordinates (x_1, \dots, x_n) , the kinetic energy has the form $T(\dot{x}) = \frac{1}{2} \sum g_{ij}(x) \dot{x}_i \dot{x}_j$ where $g_{ij}(x)$ are components of metric tensor, and the corresponding equations are

$$\ddot{x}_i + \sum_{j,l=1}^n \Gamma_{jl}^i(x) \dot{x}_j \dot{x}_l = F_i(t\omega, x, \dot{x}), \quad i \in \{1, \dots, n\},$$

where $\Gamma_{jl}^i(x)$ are the Christoffel symbols.)

In many cases the dependence of the resultant force upon velocity \dot{y} is weak and linear. For this reason it is naturally to consider that the generalize force has the form

$$F(t\omega, x, \dot{x}) = f(t\omega, x) + P(t\omega, x)\dot{x}$$

where $P(\varphi, x)$, in some sense, is small uniformly with respect to $(\varphi, x) \in \mathbb{T}^k \times \mathcal{M}$.

A classical problem for Systems (1.1) and (1.2) is whether there exists a quasiperiodic solution with frequency vector ω , i.e. a solution represented in the form $x(t) \equiv u(t\omega)$, where $u(\cdot) : \mathbb{T}^k \mapsto \mathcal{M}$ is a continuous mapping. Such a solution will be called ω -quasiperiodic.

In Euclidean configuration space with constant metric tensor, the above mentioned problem was studied by many authors even in more general almost periodic case. Non-local existence results for bounded and almost periodic solutions were obtained under certain monotonicity, convexity or coercivity conditions using topological principles, methods of nonlinear analysis, variational approach etc. (see. [4–13, 18, 19, 22, 27, 28]). A detailed enough survey on the problem can be found, e.g., in [13].

The attempts to extend results of the above papers to systems on Riemannian manifolds meet essential difficulties, especially in the case of manifolds where sectional curvature can take positive values. A number of results in this direction were obtained in [24, 25, 29] by means of variational approach. All these results concern natural Lagrangian systems. The Lagrangian density of natural time-quasiperiodic mechanical system on \mathcal{M} is represented as the difference of kinetic and potential energy: $L = \langle \dot{x}, \dot{x} \rangle / 2 - \Pi(t\omega, x)$, where $\Pi(\cdot, \cdot) : \mathbb{T}^k \times \mathcal{M} \mapsto \mathbb{R}$. The corresponding equations of motion has the form (1.1) where for any $\varphi \in \mathbb{T}^k$ the vector field $f(\varphi, \cdot)$ is the gradient of the function $-\Pi(\varphi, \cdot) : \mathcal{M} \mapsto \mathbb{R}$.

In the present paper, we obtain a novel results concerning the existence of bounded as well as quasiperiodic solutions to Systems (1.1) and (1.2). Analogously to the papers [24, 25], the corresponding existence theorems are formulated in terms of an auxiliary function $U(\cdot)$. In particular, by means of this function we introduce the notion of U -monotonicity for System (1.1). The U -monotonicity property of the system implies that the associated variational system with respect to any bounded solution is hyperbolic. Our results can be regarded as a generalization of those established in [24, 25]. In contrary to these papers, now we do not assume $f(\varphi, \cdot)$ necessarily to be the gradient of a function. Besides, we exploit a version of Ważewski topological principle instead of variational approach. In such a way

we avoid a cumbersome procedure of transition from generalized quasiperiodic solutions to classical ones. It should be noted that due to the tools of global Riemannian geometry we nowhere resorted to the usage of local coordinates.

The present paper is organized as follows. In Section 2, we formulate our main results, concerning the following issues: (a) the existence of a solution taking values in a given bounded domain of configuration space and possessing a bounded derivative; (b) the hyperbolicity of such a solution; (c) the uniqueness and, as a consequence, the quasiperiodicity of such a solution. In Section 3, a number of important auxiliary propositions are proved, including the Landau type inequality for the derivative of the bounded solution. The main theorems are proved in Section 4. In particular, here we present an ad hoc proof of quasiperiodicity without referring to the well known Amerio theorem [1]. Finally, in Section 5, the results obtained are applied to establish conditions under which the system governing the motion of charged particle in time-quasiperiodic electric field has a hyperbolic quasiperiodic solution. We also show that the perturbation of the system by sufficiently small time-quasiperiodic magnetic field together with the force of friction does not destroy such a quasiperiodic solution. The admissible magnitude of perturbation is estimated.

2. NOTATIONS AND MAIN RESULTS

In what follows we shall use the following notations: \mathcal{F} is the space of smooth (i.e. infinitely differentiable) real-valued functions on \mathcal{M} ; $T_x\mathcal{M}$ is the tangent space at the point $x \in \mathcal{M}$; \mathcal{T} is the space of smooth vector fields on \mathcal{M} ; $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$ is the norm defined by \mathbf{g} ; $\nabla_\xi v(x)$ is the covariant derivative of a vector field $v(\cdot) \in \mathcal{T}$ along a tangent vector ξ at point $x = \pi(\xi)$; for any fixed $\varphi \in \mathbb{T}^k$, $\nabla_\xi f(\varphi, x)$ and $\nabla_\xi P(\varphi, x)$ are, respectively, the covariant derivatives of tensor fields $f(\varphi, \cdot)$ and $P(\varphi, \cdot)$ along $\xi \in T_x\mathcal{M}$; $\nabla f(\varphi, \cdot)$ and $\nabla P(\varphi, \cdot)$ are, respectively, $(1, 1)$ - and $(2, 1)$ -tensor fields such that $\nabla f(\varphi, x)\xi = \nabla_\xi f(\varphi, x)$, $\nabla P(\varphi, x)(\xi, \eta) = \nabla_\xi P(\varphi, x)\eta$ for any $\xi, \eta \in T_x\mathcal{M}$; $\nabla U(x) \in T_x\mathcal{M}$ and $H_U(x) : T_x\mathcal{M} \mapsto T_x\mathcal{M}$ are, respectively, the gradient vector and the Hesse form at x of a function $U(\cdot) \in \mathcal{F}$ (by the definition $\langle H_U(x)\xi, \eta \rangle = \langle \nabla_\xi \nabla U(x), \eta \rangle$ for any $x \in \mathcal{M}$ and any $\xi, \eta \in T_x\mathcal{M}$); if $W(\cdot, \cdot) : \mathbb{T}^k \times \mathcal{M} \mapsto \mathbb{R}$ is a smooth function, then for any fixed $\varphi \in \mathbb{T}$ the function $W(\varphi, \cdot) \in \mathcal{F}$ naturally defines the gradient $\nabla W(\varphi, x)$ and the Hesse form $H_W(\varphi, x)$ at the point x .

Let R be the curvature tensor of Levi – Civita connection (defined as in [14]) $\sigma = \sigma(\xi, \eta)$ be a 2-dimensional plane spanned on linearly independent vectors $\xi, \eta \in T_x\mathcal{M}$. Then

$$K_\sigma(x) := \frac{\langle R(\xi, \eta)\eta, \xi \rangle}{\|\xi\|^2 \|\eta\|^2 - \langle \xi, \eta \rangle^2}$$

is the Riemannian curvature in direction σ at the point $x \in \mathcal{M}$ (see, e.g. [14]). Denote by \mathfrak{G}_x^2 the Grassmann manifold of 2-dimensional linear subspaces in $T_x\mathcal{M}$ and define

$$K(x) := \max \{0, \sup \{K_\sigma(x) : \sigma \in \mathfrak{G}_x^2\}\}.$$

Set

$$M_f(x) := \max \{\|f(\varphi, x)\| : \varphi \in \mathbb{T}^k\}, \quad (2.1)$$

$$\Lambda_P(x) := \max \{\langle P(\varphi, x)\xi, \xi \rangle : \varphi \in \mathbb{T}^k, \xi \in T_x\mathcal{M}, \|\xi\| = 1\} \quad (2.2)$$

Now let us formulate the results concerning the existence of bounded solutions to Systems (1.1) and (1.2).

Theorem 1. *Let the following hypotheses be satisfied:*

H1: *there exist a function $U(\cdot) \in \mathcal{F}$ and a bounded domain $\mathcal{D} \subset \mathcal{M}$ such that*

$$\lambda_U(x) := \min \{\langle H_U(x)\xi, \xi \rangle : \xi \in T_x\mathcal{M}, \|\xi\| = 1\} > 0 \quad \forall x \in \text{cl}(\mathcal{D}) \quad (2.3)$$

and

$$\min \{ \langle \nabla U(x), f(\varphi, x) \rangle : (\varphi, x) \in \mathbb{T}^k \times \text{cl}(\mathcal{D}) \} < 0;$$

H2: the boundary $\partial\mathcal{D}$ of the domain \mathcal{D} is a smooth hypersurface and for any $(\varphi, x) \in \mathbb{T}^k \times \partial\mathcal{D}$ there hold the inequalities

$$\langle \nu(x), f(\varphi, x) \rangle > 0, \quad \lambda_{II}(x) > 0$$

where $\nu(x)$ and $\lambda_{II}(x)$ stand, respectively, for the unite vector of outward normal and the minimal principal curvature of the boundary¹ at point $x \in \partial\mathcal{D}$, i.e.

$$\lambda_{II}(x) := \min \{ \langle \nabla_\xi \nu(x), \xi \rangle : \xi \in T_x \partial\mathcal{M}, \|\xi\| = 1 \}.$$

Then System(1.2) has a solution $x_*(\cdot) : \mathbb{R} \mapsto \mathcal{D}$ such that

$$\sup_{t \in \mathbb{R}} \|\dot{x}_*(t)\| \leq z_* := q\zeta_*(C_f C_U / q^2)$$

where

$$C_f := \max \left\{ \frac{M_f(x)}{\lambda_U(x)} : x \in \text{cl}(\mathcal{D}) \right\}, \quad C_U := \max \{ \|\nabla U(x)\| : x \in \text{cl}(\mathcal{D}) \}, \quad (2.4)$$

$$q := \sqrt{\max \left\{ -\frac{\langle \nabla U(x), f(\varphi, x) \rangle}{\lambda_U(x)} : (\varphi, x) \in \mathbb{T}^k \times \text{cl}(\mathcal{D}) \right\}}, \quad (2.5)$$

and $\zeta_*(m)$ stands for the greatest root of the polynomial $\zeta \mapsto \zeta^3 - 3\zeta + 2 - 3m$.

Remark 1. If $\lambda_U(x) > 0$ and $\langle \nabla U(x), f(\varphi, x) \rangle$ is non-negative in $\mathbb{T}^k \times \text{cl}(\mathcal{D})$, then System (1.1) does not have non-constant bounded on \mathbb{R}_+ solutions (see Remark 6 below).

The proof of Theorem 1 remains correct if instead of (2.5) we put $q = \sqrt{C_f C_U}$. Since the greatest root of the polynomial $\zeta^3 - 3\zeta - 1$ does not exceed 1.88, then the above estimate for $\|\dot{x}_*(t)\|$ can be replaced by the following one

$$\sup_{t \in \mathbb{R}} \|\dot{x}_*(t)\| \leq 1.88 \sqrt{C_f C_U}.$$

Remark 2. In [13], for a system $\ddot{x} = f(t\omega, x)$ in Euclidean space \mathbb{E}^n an estimate for derivative of solution $x(\cdot) : \mathbb{R} \mapsto \mathcal{B}_R := \{x : \|x\| \leq R\}$ is obtained by means of the Landau inequality. With the Hadamard best possible constant, this inequality reads as follows

$$\sup_{t \in \mathbb{R}} \|\dot{x}(t)\| \leq \sqrt{2 \sup_{t \in \mathbb{R}} \|x(t)\| \sup_{t \in \mathbb{R}} \|\ddot{x}(t)\|}.$$

If we take $U(x) := \|x\|^2/2$, then $C_U = R$, $\lambda_U = 1$, and

$$\sup_{t \in \mathbb{R}} \|\ddot{x}(t)\| \leq \max \{ \|f(\varphi, x)\| : \varphi \in \mathbb{T}^k, \|x\| \leq R \} = C_f.$$

Thus, in the case of $\mathcal{M} = \mathbb{E}^n$, the Landau-Hadamard inequality yields somewhat better estimate $\sup_{t \in \mathbb{R}} \|\dot{x}(t)\| \leq \sqrt{2C_f C_U}$.

Now let us proceed to the perturbed system. Set

$$l := \max \left\{ \frac{\Lambda_P(x)}{M_f(x)} : x \in \text{cl}(\mathcal{D}) \right\}, \quad (2.6)$$

$$p := \max \left\{ \frac{\|P^*(\varphi, x) \nabla U(x)\|}{\lambda_U(x)} : (\varphi, x) \in \mathbb{T}^k \times \text{cl}(\mathcal{D}) \right\}$$

¹Here the second fundamental tensor for $\partial\mathcal{D}$ with respect to inward normal vector field is defined in the following way: $\langle II(x)\xi, \eta \rangle = \langle \nabla_\xi \nu(x), \eta \rangle \quad \forall \xi, \eta \in T_x \partial\mathcal{M}$.

where the conjugate $P^*(\varphi, x)$ is defined in a standard way: $\langle P(\varphi, x)\xi, \eta \rangle = \langle \xi, P^*(\varphi, x)\eta \rangle$ for all $\xi, \eta \in T_x\mathcal{M}$. Introduce the numbers

$$z_{\pm} := \frac{p \pm \sqrt{4q^2 + p^2}}{2} \quad (2.7)$$

(the roots of the polynomial $z \mapsto z^2 - pw - q^2$) and the function

$$I(z) := \int_{z_+}^z \frac{(w^2 - pw - q^2)}{lw + 1} dw = \frac{1}{l} \left[\frac{1}{2}w^2 - \frac{1+pl}{l}w + \frac{1+pl-q^2l^2}{l^2} \ln(1+lw) \right] \Big|_{z_+}^z.$$

Theorem 2. *Let Hypotheses **H1** and **H2** be satisfied, and in addition,*

$$4\lambda_{II}(x) \langle \nu(x), f(\varphi, x) \rangle > \|P^*(\varphi, x)\nu(x)\|^2 \quad \forall (\varphi, x) \in \mathbb{T}^k \times \partial\mathcal{D}. \quad (2.8)$$

Then System(1.2) has a solution $x_(\cdot) : \mathbb{R} \mapsto \mathcal{D}$ such that $\sup_{t \in \mathbb{R}} \|\dot{x}_*(t)\|$ does not exceed the root $z_* \in [z_+, \infty)$ of equation $I(z) = C_f C_U z_+$.*

Remark 3. Suppose that $l < z_+/q^2$, and thus $1 + pl - q^2l^2 > 0$. Since $I(z_+) = I'(z_+) = 0$, $I'''(z) > 0$ and $I''(z_+) > 2q(1 + lz_+)^{-2}$, then

$$I(z) \geq \frac{1}{2}I''(z_+)(z - z_+)^2 \geq \frac{q(z - z_+)^2}{(1 + lz_+)^2},$$

and hence

$$z_* \leq z_+ + (1 + lz_+) \sqrt{\frac{C_f C_U z_+}{q}}.$$

The next two results establish conditions which ensures the hyperbolicity of bounded solutions. Recall the corresponding notion. Let $x(\cdot, \cdot) : (t_1, t_2) \times (-\sigma, \sigma) \mapsto \mathcal{M}$ be such a smooth mapping that $x(\cdot, s) : (t_1, t_2) \mapsto \mathcal{M}$ is a solution of (1.2) for any fixed $s \in (-\sigma, \sigma)$, the number $\sigma > 0$ being sufficiently small. Define two tangent vector fields along the mapping $x(\cdot, \cdot)$:

$$\dot{x}(t, s) := \frac{\partial x(t, s)}{\partial t}, \quad x'(t, s) := \frac{\partial x(t, s)}{\partial s}.$$

Then $\nabla_{\dot{x}} x' = \nabla_{x'} \dot{x}$ and

$$\nabla_{x'} \nabla_{\dot{x}} \dot{x} - \nabla_{\dot{x}} \nabla_{x'} \dot{x} = R(x', \dot{x}) \dot{x}.$$

Since

$$\nabla_{x'} \nabla_{\dot{x}} \dot{x} = \nabla_{x'} (f(t\omega, x) + P(t\omega, x) \dot{x})$$

then

$$\nabla_{\dot{x}}^2 x' = \nabla f(t, x) x' - R(x', \dot{x}) \dot{x} + \nabla P(t\omega, x)(x', \dot{x}) + P(t\omega, x) \nabla_{\dot{x}} x'.$$

Put here $s = 0$ and denote $\tau(t) := \dot{x}(t, 0)$. The vector fields

$$\eta(t) := x'(t, 0), \quad \zeta(t) = \nabla_{\tau} \eta(t),$$

along the mapping $x(\cdot) := x(\cdot, 0)$ satisfy the first order linear variational system with respect to solution $x(t)$:

$$\nabla_{\tau(t)}\eta = \zeta$$

$$\nabla_{\tau(t)}\zeta = [\nabla f(t\omega, x)\eta - R(\eta, \tau(t))\tau(t) + \nabla P(t\omega, x)(\eta, \tau(t)) + P(t\omega, x)\zeta]_{x=x(t)}.$$

If the solution $x(t)$ is extendable on the whole real axis and the variational system is exponentially dichotomic on \mathbb{R} , then such a solution is called hyperbolic.

Definition 1. We shall say that System (1.1) is U -monotone in \mathcal{D} if there exists $U(\cdot) \in \mathcal{F}$ satisfying the inequalities

$$\lambda_f(\varphi, x) + \frac{\langle \nabla U(x), f(\varphi, x) \rangle}{2} > 0 \quad \forall (\varphi, x) \in \mathbb{T}^k \times \text{cl}(\mathcal{D}), \quad (2.9)$$

$$\mu_U(x) \geq 2K(x) \quad \forall x \in \mathcal{D} \quad (2.10)$$

where

$$\lambda_f(\varphi, x) := \min \{ \langle \nabla f(\varphi, x)\eta, \eta \rangle : \eta \in T_x\mathcal{M}, \|\eta\| = 1 \},$$

$$\mu_U(x) := \min \left\{ \langle H_U(x)\eta, \eta \rangle - \frac{\langle \nabla U(x), \eta \rangle^2}{2} : \eta \in T_x\mathcal{M}, \|\eta\| = 1 \right\}. \quad (2.11)$$

Remark 4. Let $\mathcal{M} = \mathbb{E}^n$, $K(x) \equiv 0$. The standard monotonicity condition for a second-order system $\ddot{x} = f(t\omega, x)$ requires the quadratic form $y \mapsto \langle f'_x(\varphi, x)y, y \rangle$ to be positive definite. In such a case, if $x(\cdot)$ is a solution of the second order system, then the indefinite quadratic form $\langle y, z \rangle$ in $\mathbb{R}^n \times \mathbb{R}^n$ has a positive definite derivative along any solution of variational system $\dot{y} = z$, $\dot{z} = f'_x(t\omega, x(t))y$ equivalent to second order linear system $\ddot{y} = f'_x(t\omega, x(t))y$, and thus the variational system is dichotomic [26]. If \mathcal{M} is such a manifold that $K(x) > 0$, then one can try to ensure the U -monotonicity by means of appropriate choice of function $U(\cdot)$. As will be shown below, if (1.1) is U -monotone, then the modified indefinite non-degenerate quadratic form

$$(\eta, \zeta) \mapsto \langle \eta, \zeta \rangle + \frac{\langle \nabla U(x(t)), \dot{x}(t) \rangle \|\eta\|^2}{2}$$

has positive derivative along solutions of variational system

$$\nabla_{\tau(t)}\eta = \zeta,$$

$$\nabla_{\tau(t)}\zeta = [\nabla f(t\omega, x)\eta - R(\eta, \tau(t))\tau(t)]_{x=x(t)}.$$

Thus, this system is hyperbolic.

Theorem 3. *Let the following hypothesis be satisfied*

H3: *System (1.1) is U -monotone in \mathcal{D} .*

If $x(\cdot) : \mathbb{R} \mapsto \mathcal{D}$ is a solution of System (1.1) such that $\sup_{t \in \mathbb{R}} \|\dot{x}(t)\| < \infty$, then this solution is hyperbolic.

The next theorem concerns the perturbed system. It is well known that sufficiently small perturbations do not destroy the hyperbolic solution of unperturbed system. With our approach we are able to establish realistic bounds for perturbations which preserve the hyperbolicity of solution contained in \mathcal{D} .

Set

$$M_P(\varphi, x) := \|P(\varphi, x)\|, \quad M_U(x) := \|\nabla U(x)\|, \quad M_{PU}(\varphi, x) := \|P^*(\varphi, x)\nabla U(x)\|, \quad (2.12)$$

$$L_P(\varphi, x) := \max \{ |\langle \nabla P(\varphi, x)(\eta, \xi), \eta \rangle| : \xi, \eta \in T_x\mathcal{M}, \|\xi\| = \|\eta\| = 1 \}. \quad (2.13)$$

Theorem 4. *Let Hypothesis H3 be satisfied. If $x(\cdot) : \mathbb{R} \mapsto \mathcal{D}$ is a solution of System (1.2) such that $\sup_{t \in \mathbb{R}} \|\dot{x}(t)\| := Z < \infty$, and in addition,*

$$\lambda_f(\varphi, x) + \frac{\langle \nabla U(x), f(\varphi, x) \rangle}{2} > \sigma(\varphi, x; Z) \quad \forall (\varphi, x) \in \mathbb{T}^k \times \text{cl}(\mathcal{D}) \quad (2.14)$$

where

$$\sigma(\varphi, x; Z) := \frac{(M_U(x)M_P(\varphi, x) + M_{PU}(\varphi, x) + 2L_P(\varphi, x))Z}{2} + \frac{M_P^2(\varphi, x)}{4},$$

then the solution $x(t)$ is hyperbolic.

Finally, let us present the results on the existence of quasiperiodic solutions.

Theorem 5. *Let the Hypotheses **H1** – **H3** be satisfied, and in addition, suppose that there holds the inequality*

$$\lambda_{II}(x) + \frac{1}{2} \langle \nabla U(x), \nu(x) \rangle > 0 \quad \forall x \in \partial \mathcal{D}. \quad (2.15)$$

Then the domain \mathcal{D} contains the unique solution $x_*(\cdot) : \mathbb{R} \mapsto \mathcal{D}$ of System (1.1). This solution is ω -quasiperiodic and hyperbolic.

Remark 5. Suppose that there exists a noncritical value $c \in U(\mathcal{M})$ such that $\lambda_U(x) > 0$ in a connected component $\tilde{\mathcal{D}}$ of sub-level set $U^{-1}(-\infty, c)$. Then the inequality (2.15) is satisfied for all $x \in \partial \tilde{\mathcal{D}}$.

For the perturbed system. the corresponding statement is as follows.

Theorem 6. *Let the Hypotheses **H1** – **H3** be satisfied, and in addition, suppose that there holds the inequalities (2.15), (2.8) and*

$$\lambda_f(\varphi, x) + \frac{\langle \nabla U(x), f(\varphi, x) \rangle}{2} > \sigma(\varphi, x; z_*) \quad \forall (\varphi, x) \in \mathbb{T}^k \times \text{cl}(\mathcal{D}) \quad (2.16)$$

where z_* and $\sigma(\cdot, \cdot; \cdot)$ are defined, respectively, in Theorem 2 and Theorem 4. Then the domain \mathcal{D} contains the unique solution $x_*(\cdot) : \mathbb{R} \mapsto \mathcal{D}$ of System (1.2). This solution is ω -quasiperiodic and hyperbolic.

3. AUXILIARY PROPOSITIONS

Propositions 1 – 2 below are essential for the proof of Theorems 1 and 2.

Proposition 1. *Let $\mathcal{D} \subset \mathcal{M}$ be a bounded domain and let $x(\cdot) : (T_-, T_+) \mapsto \mathcal{M}$ be a non-extendable solution of System (1.2) such that $x(\cdot) : [s, T_+) \mapsto \text{cl}(\mathcal{D})$ for some $s \in (T_-, T_+)$. Then $T_+ = \infty$. If $x(\cdot) : (T_-, T_+) \mapsto \text{cl}(\mathcal{D})$ then $(T_-, T_+) = \mathbb{R}$.*

Proof. If we assume that $T_+ < \infty$, then $\limsup_{t \nearrow T_+} \|\dot{x}(t)\| = \infty$. Since

$$\frac{d}{dt} \|\dot{x}(t)\|^2 = 2 \langle \dot{x}, f(t\omega, x) + P(\omega t, x)\dot{x} \rangle \Big|_{x=x(t)} \leq [1 + 2\Lambda_P(x(t))] \|\dot{x}(t)\|^2 + M_f^2(x(t)),$$

then $\|\dot{x}(t)\|^2$ does not exceed the solution of linear initial problem

$$\dot{z} = [1 + 2\Lambda_P(x(t))] z + M_f^2(x(t)), \quad z(s) = \|\dot{x}(s)\|^2.$$

Hence $\|\dot{x}(t)\|$ is bounded on $[s, T_+)$ and we arrive at contradiction with our assumption that $T_+ < +\infty$.

The same arguments can be used to prove that $T_- = -\infty$ if $x(\cdot) : (-\infty, s] \mapsto \text{cl}(\mathcal{D})$. \square

If $\mathcal{M} = \mathbb{R}^n$, and the system is $\ddot{x} = f(t\omega, x)$, then the boundedness of solution on \mathbb{R}_+ or \mathbb{R} implies the boundedness of its second derivative, respectively, on \mathbb{R}_+ or \mathbb{R} . In [13], the Landau inequality was used to prove the boundedness of the first derivative of solution. However, in the case of Riemannian manifold with non-constant metric tensor, the equation (1.2) written in local coordinates contains quadratic terms with respect to \dot{x} . Thus the Landau inequality cannot be directly applied to prove the boundedness of $\dot{x}(\cdot)$. Nevertheless, we have

Proposition 2. *Suppose that Hypothesis **H1** is valid and let $x(\cdot) : [s, \infty) \mapsto \text{cl}(\mathcal{D})$ be a solution of System (1.2) such that $\|\dot{x}(s)\| \leq z_+$ (see (2.7)). Then $\sup_{t \geq s} \|\dot{x}(t)\| \leq z_*$ where z_* is defined in Theorem 1 if $P(\varphi, x) \equiv 0$, and in Theorem 2 otherwise.*

Proof. Define $u(t) := U(x(t))$, $v(t) := \dot{u}(t) \equiv \langle \nabla U(x(t)), \dot{x}(t) \rangle$. We have

$$\begin{aligned} \dot{v}(t) &= [\langle H_U(x) \dot{x}, \dot{x} \rangle + \langle \nabla U(x), f(t\omega, x) + P(t\omega, x) \dot{x} \rangle]_{x=x(t)} \geq \\ &= [\lambda_U(x) \|\dot{x}\|^2 - \|P^*(t\omega, x) \nabla U(x)\| \|\dot{x}(t)\| + \langle \nabla U(x), f(t\omega, x) \rangle]_{x=x(t)} \geq \\ &= [\lambda_U(x) (\|\dot{x}\|^2 - p \|\dot{x}\| - q^2)]_{x=x(t)} = [\lambda_U(x) (\|\dot{x}\| - z_-) (\|\dot{x}\| - z_+)]_{x=x(t)}. \end{aligned}$$

Let us show that $|v(t)| \leq C_U z_+$ for all $t \geq s$. By reasoning ad absurdum, suppose that for some $\delta > 0$ the set

$$\mathcal{T}_{v,\delta} := \{t \geq s : |v(t)| > C_U(z_+ + \delta)\}$$

is non-empty. Since $|v(t)| \leq C_U \|\dot{x}(t)\|$, then $\|\dot{x}(t)\| > z_+ + \delta$ for all $t \in \mathcal{T}_{v,\delta}$. Hence,

$$\dot{v}(t) \geq [\lambda_U(x) (z_+ + \delta - \bar{z}_-) \delta]_{x=x(t)} \geq l_U \delta^2 > 0 \quad \forall t \in \mathcal{T}_{v,\delta}$$

where

$$l_U := \min_{x \in \text{cl}(\mathcal{D})} \lambda_U(x). \quad (3.1)$$

Thus $v(t)$ does not decrease while $t \in \mathcal{T}_{v,0}$. Since $|v(s)| \leq C_U \|\dot{x}(s)\| \leq C_U z_+$, then $s \notin \mathcal{T}_{v,\delta}$ and for this reason $v(t) \geq -C_U z_+$ for all $t \geq s$. On the other hand, if $t_0 \in \mathcal{T}_{v,\delta}$ then

$$v(t) \geq v(t_0) + l_U \delta^2 (t - t_0) > C_U(z_+ + \delta)$$

while $t > t_0$ and $t \in \mathcal{T}_{v,\delta}$. This implies that $[t_0, \infty) \subset \mathcal{T}_{v,\delta}$, $v(t) \rightarrow \infty$ as $t \rightarrow \infty$, and hence, $u(t) \rightarrow \infty$ as $t \rightarrow \infty$. We arrive at contradiction with our assumption that $x(t) \in \mathcal{D}$ for all $t \geq s$, since $U(\cdot)$ is bounded in $\text{cl } \mathcal{D}$.

Now let us estimate $\|\dot{x}(t)\|$. Consider the nontrivial case where the set

$$\mathcal{T} := \{t > s : \|\dot{x}(t)\| > z_+\}$$

is non-empty. Obviously that any connected component of this set is an interval (t_1, t_2) such that $t_1 \geq s$, $t_2 \leq +\infty$, and $\|\dot{x}(t_1)\| = z_+$; besides, $\|\dot{x}(t_2)\| = z_+$ if $t_2 < +\infty$, and $\liminf_{t \rightarrow +\infty} \|\dot{x}(t)\| = z_+$ if $t_2 = \infty$. In fact, if the last equality were wrong, then the same arguments as above would lead to unboundedness of $v(t)$.

Since for any $t \in (t_1, t_2)$ we have

$$\begin{aligned} \left| \frac{d}{dt} \|\dot{x}(t)\|^2 = 2 \langle \dot{x}, f(t\omega, x) + P(t\omega, x) \dot{x} \rangle \right|_{x=x(t)} &\leq 2 [M_f(x) \|\dot{x}\| + \Lambda_P(x) \|\dot{x}\|^2]_{x=x(t)} \Rightarrow \\ \left| \frac{\frac{d}{dt} \|\dot{x}\|}{M_f(x) + \Lambda_P(x) \|\dot{x}\|} \right|_{x=x(t)} &\leq 1, \end{aligned}$$

(see (2.1), (2.2)), and

$$\dot{v}(t) \geq [\lambda_U(x) (\|\dot{x}\|^2 - p \|\dot{x}\| - q^2)]_{x=x(t)} > 0,$$

then

$$\left| \frac{[\lambda_U(x) (\|\dot{x}\|^2 - p \|\dot{x}\| - q^2)] \frac{d}{dt} \|\dot{x}\|}{M_f(x) + \Lambda_P(x) \|\dot{x}\|} \right|_{x=x(t)} \leq \dot{v}(t),$$

and finally,

$$\left| \frac{(\|\dot{x}\|^2 - p \|\dot{x}\| - q^2) \frac{d}{dt} \|\dot{x}\|}{1 + l \|\dot{x}\|} \right|_{x=x(t)} \leq C_f \dot{v}(t) \quad \forall t \in (t_1, t_2)$$

(see (2.4), (2.6)). This yields

$$-\dot{v}(t) \leq \frac{d}{dt} I(\|\dot{x}(t)\|) \leq \dot{v}(t) \quad \forall t \in (t_1, t_2).$$

Then for any $t \in (t_1, t_2)$ and for any sufficiently small $\varepsilon > 0$ there exists $t_\varepsilon \in (t, t_2)$ such that $\|\dot{x}(t_\varepsilon)\| = z_+ + \varepsilon$. Now

$$\begin{aligned} 2C_f C_U z_+ + &\geq v(t_\varepsilon) - v(t_1) = \int_{t_1}^t \dot{v}(s) ds + \int_t^{t_\varepsilon} \dot{v}(s) ds \geq \\ &\geq \int_{t_1}^t \frac{d}{ds} I(\|\dot{x}(s)\|) ds - \int_t^{t_\varepsilon} \frac{d}{ds} I(\|\dot{x}(s)\|) ds \geq 2I(\|\dot{x}(t)\|) - I(z_+ + \varepsilon). \end{aligned}$$

Letting $\varepsilon \rightarrow +0$ we obtain $I(\|\dot{x}(t)\|) \leq C_f C_U z_+$ for all $t \in (t_1, t_2)$. Since $I(z_+) = 0$ and $I(z)$ monotonically tends to $+\infty$ on $[z_+, +\infty)$, then there exists a unique $z_* > z_+$ such that $I(z_*) = C_f C_U z_+$. This implies the required estimate for $\|\dot{x}(t)\|$.

In the case where $P(\varphi, x) \equiv 0$, we have $l = 0$, $p = 0$, $z_+ = -z_- = q$, and

$$I(z) = \frac{z^3}{3} - q^2 z + \frac{2q^3}{3}.$$

Hence, z_* is a solution of equation

$$z^3 - 3q^2 z + 2q^3 = 3C_f C_U q.$$

After the substitution $z = q\zeta$, $m = C_f C_U / q^2$ we obtain the equation

$$\zeta^3 - 3\zeta + 2 = 3m.$$

□

Remark 6. If $P(\varphi, x) \equiv 0$ and $\langle \nabla U(x), f(\varphi, x) \rangle$ is non-negative in $\mathbb{T}^k \times \text{cl}(\mathcal{D})$, then from the inequality $\dot{v}(t) \geq l_U \|\dot{x}\|^2 \geq v(t)/C_U$ it follows that System (1.1) does not have non-constant solutions $x(\cdot) : [s, \infty) \mapsto \text{cl}(\mathcal{D})$.

Let $U(\cdot) \in \mathcal{F}$. Consider the initial value problem

$$\nabla_{x'} x' = \frac{\|x'\|^2}{2} \nabla U(x), \quad x(0) = x_0 := \pi(\xi), \quad x'(0) = \xi \in T\mathcal{M} \quad \left(x' = \frac{dx}{ds} \right). \quad (3.2)$$

Propositions 3 and 4 below are essentially exploited in the proof of Theorems 5 and 6.

Proposition 3. *Let \mathcal{D} be a domain in \mathcal{M} , $x_0 \in \mathcal{D}$, $\xi_0 \in T_{x_0}\mathcal{M}$, and let $x(\cdot, \xi) : [0, 1] \mapsto \text{cl}(\mathcal{D})$ be the family of solutions to initial value problem (3.2) with parameter ξ ranging in a neighborhood of vector ξ_0 . Suppose that the function $U(\cdot)$ satisfies in \mathcal{D} the inequality (2.10). Then the derivative of $x(s, \cdot)$ along $T_{x_0}\mathcal{M}$,*

$$\frac{\partial}{\partial \xi} \Big|_{\xi=\xi_0} x(s, \xi) : T_{\xi_0}(T_{x_0}\mathcal{M}) \cong T_{x_0}\mathcal{M} \mapsto T_{x(s, \xi_0)}\mathcal{M},$$

is non-degenerate for all $s \in (0, 1]^2$

²Recall the well-known fact from the theory of ODE (see, e.g. [15]): if we denote by $I(\xi)$ the interval of existence for non-extendable solution to (3.2), then the set $\mathcal{E} := \{(s, \xi) : \xi \in T\mathcal{M}, s \in I(\xi)\}$ is open in $\mathbb{R} \times T\mathcal{M}$ and the mapping $x(\cdot, \cdot) : \mathcal{E} \mapsto \mathcal{M}$ is smooth.

Proof. For a given smooth curve $\xi(\cdot) : (-\sigma, \sigma) \mapsto T_{x_0}\mathcal{M}$, where $\sigma > 0$ and $\xi(0) = \xi_0$, construct the mapping

$$x(\cdot, \xi(\cdot)) : I \times (-\sigma, \sigma) \mapsto \mathcal{M}.$$

Define the following two tangent vector fields along this mapping

$$Y(s, r) := \frac{\partial x(s, \xi(r))}{\partial r}, \quad Z(s, r) = \frac{\partial x(s, \xi(r))}{\partial s}.$$

Then $\nabla_Y Z = \nabla_Z Y$ and

$$\nabla_Y \nabla_Z Z - \nabla_Z \nabla_Y Z = R(Y, Z)Z.$$

Since

$$\nabla_Y \nabla_Z Z = \nabla_Y \left(\frac{\|Z\|^2}{2} \nabla U(x) \right),$$

then

$$\nabla_Z^2 Y = \langle \nabla_Y Z, Z \rangle \nabla U(x) + \frac{\|Z\|^2}{2} H_U(x) Y - R(Y, Z)Z.$$

Put here $r = 0$ and denote $\bar{x}(s) := x(s, \xi_0)$, $\tau(s) := Z(s, 0) \equiv \bar{x}'(s)$. We see that the vector fields

$$\eta(s) := Y(s, 0), \quad \zeta(s) = \nabla_\tau \eta(s),$$

along the mapping $\bar{x}(\cdot)$ satisfy the first order system in variations:

$$\begin{aligned} \nabla_\tau \eta &= \zeta, \\ \nabla_\tau \zeta &= \langle \zeta, \tau \rangle \nabla U + \frac{\|\tau\|^2}{2} H_U(\bar{x}) \eta - R(\eta, \tau) \tau \end{aligned}$$

We have to show that $\eta(s) \neq 0$ for all $s \in (0, 1]$ once $\xi'(0) \neq 0$. Since

$$\frac{d}{ds} \frac{\|\eta\|^2}{2} = \langle \nabla_\tau \eta, \eta \rangle = \langle \eta, \zeta \rangle$$

then

$$\begin{aligned} \frac{d^2}{ds^2} \frac{\|\eta\|^2}{2} &= \frac{d}{ds} \langle \eta, \zeta \rangle = \langle \zeta, \zeta \rangle + \left\langle \eta, \langle \zeta, \tau \rangle \nabla U(\bar{x}) + \frac{\|\tau\|^2}{2} H_U(\bar{x}) \eta - R(\eta, \tau) \tau \right\rangle = \\ &= \|\zeta\|^2 + \langle \nabla U(\bar{x}), \eta \rangle \langle \zeta, \tau \rangle + \frac{\|\tau\|^2}{2} \langle H_U(\bar{x}) \eta, \eta \rangle - K_{\sigma(\eta, \tau)}(\bar{x}) [\|\eta\|^2 \|\tau\|^2 - \langle \eta, \tau \rangle^2] \geq \\ &= \|\zeta\|^2 - \|\zeta\| \|\tau\| \langle \nabla U(\bar{x}), \eta \rangle + \|\tau\|^2 \left[\frac{\langle H_U(\bar{x}) \eta, \eta \rangle}{2} - K(\bar{x}) \|\eta\|^2 \right]. \end{aligned}$$

The condition (2.10) yields that $\frac{d^2}{ds^2} \|\eta(s)\|^2 \geq 0$.

Since

$$x(0, \xi(r)) = x_0, \quad \frac{\partial}{\partial s} \Big|_{s=0} x(s, \xi(r)) = \xi(r),$$

then $\eta(0) = 0$. From this it follow that the horizontal component of vector $\eta'(0)$ (with respect to the Levi-Civita connection) vanishes and we obtain

$$\zeta(0) = \nabla_\tau \eta|_{s=0} = \frac{\partial}{\partial s} \Big|_{s=0} \frac{\partial}{\partial r} \Big|_{r=0} x(s, \xi(r)) + O(\|\eta\|) = \xi'(0).$$

Hence, if $\xi'_0 := \xi'(0) \neq 0$, then

$$\|\eta(0)\|^2 = 0, \quad \frac{d}{ds} \Big|_{s=0} \|\eta(s)\|^2 = 0, \quad \frac{d^2}{ds^2} \Big|_{s=0} \|\eta(s)\|^2 = 2 \|\zeta(0)\|^2 = 2 \|\xi'_0\|^2 > 0.$$

This implies that $\|\eta(s)\| > 0$ for all $s \in (0, 1]$. □

Proposition 4. *Suppose that a function $U(\cdot) \in \mathcal{F}$ in a bounded domain \mathcal{D} with smooth boundary satisfies the inequalities (2.10), (2.15), and let $x(\cdot, \xi)$ be the solution of initial value problem (3.2). Then for any $\{x_0, x_1\} \subset \text{cl}(\mathcal{D})$ there exists $\xi(x_0, x_1) \in T_{x_0}\mathcal{M}$ such that $x(s, \xi(x_0, x_1)) \in \mathcal{D}$ for all $s \in (0, 1)$, and $x(1, \xi(x_0, x_1)) = x_1$. Moreover,*

$$\|\xi(x_0, x_1)\| \leq d \quad \forall \{x_0, x_1\} \subset \text{cl}(\mathcal{D})$$

where

$$d := \frac{C_U e^{U^* - U_*} + \sqrt{(C_U e^{U^* - U_*})^2 + 2l_U e^{U^* - U_*} (U^* - U_*)}}{l_U},$$

$$U_* = \min \{U(x) : x \in \text{cl}(\mathcal{D})\}, \quad U^* = \max \{U(x) : x \in \text{cl}(\mathcal{D})\}. \quad (3.3)$$

Proof. Let us fix $x_0 \in \mathcal{D}$ arbitrarily and define the set

$$\Xi = \{\xi \in T_{x_0}\mathcal{M} : x(s, \xi) \in \mathcal{D} \forall s \in [0, 1]\}.$$

This set is non-empty and open in $T_{x_0}\mathcal{M}$. In fact, $0 \in \Xi$ and if $\xi_0 \in \Xi$ then for all $\xi \in T_{x_0}\mathcal{M}$ sufficiently close to ξ_0 the solution $x(s, \xi)$ is defined on $[0, 1]$ and takes values in \mathcal{D} (see the footnote 2 on page 9). This means that a small neighborhood of $\xi_0 \in \Xi$ is contained in Ξ . By Proposition 3 the mapping

$$X(\cdot) := x(1, \cdot) : \Xi \mapsto \mathcal{D}$$

has non-degenerate derivative $X'(\xi)$ at each point $\xi \in \Xi$. Hence, this mapping is a local diffeomorphism and for this reason the set $\mathcal{X} := X(\Xi)$ is an open subset of \mathcal{D} .

To show that $\mathcal{X} = \mathcal{D}$ it remains to prove that the set \mathcal{X} is closed in \mathcal{D} . If we suppose the opposite to be true, then there exists a sequence $\{x_k\} \subset \mathcal{X}$ convergent to $x_* \in \mathcal{D} \setminus \mathcal{X}$. By the definition of \mathcal{X} , there also exists a sequence $\{\xi_k\} \subset \Xi$ such that $x_k = x(1, \xi_k)$.

Let us show that the sequence $\{\xi_k\}$ is bounded. Observe that for any $\xi \in \Xi$ and $s \in [0, 1]$ we have

$$\left[\frac{d}{ds} \left(\|x'\|^2 e^{-U(x)} \right) = \langle x', \nabla U(x) \rangle \|x'\|^2 - \|x'\|^2 \langle x', \nabla U(x) \rangle = 0 \right]_{x=x(s, \xi)}$$

Hence,

$$\|x'(s, \xi)\|^2 = \|\xi\|^2 \exp(U(x_0) - U(x(s, \xi))). \quad (3.4)$$

Since

$$\left[\frac{d^2}{ds^2} U(x) = \langle H_U(x) x', x' \rangle + \langle \nabla U(x), \nabla_{x'} x' \rangle = \langle H_U(x) x', x' \rangle + \frac{\|x'\|^2}{2} \|\nabla U(x)\|^2 \right]_{x=x(s, \xi)},$$

then by the Taylor formula there exists $\theta_k \in (0, 1)$ such that

$$U(x_k) = U(x_0) + \langle \nabla U(x_0), \xi_k \rangle + \frac{1}{2} \left[\langle H_U(x) x', x' \rangle + \frac{\|x'\|^2}{2} \|\nabla U(x)\|^2 \right]_{x=x(\theta_k, \xi_k)}.$$

The condition (2.3) yields

$$U(x_k) \geq U(x_0) - |\langle \nabla U(x_0), \xi_k \rangle| + \frac{l_U}{2} \|\xi_k\|^2 \exp(U(x_0) - U(x(\theta_k, \xi_k)))$$

(see (3.1)). Now obviously the sequence $\{\|\xi_k\|\}$ is bounded and without loss of generality, one can regard that $\xi_k \rightarrow \xi_* \in T_{x_0}\mathcal{M} \setminus \Xi$. Since $x_* = x(1, \xi_*) \in \mathcal{D}$, then there is $s_* \in (0, 1)$ such that $x(s, \xi_*) \in \mathcal{D}$ for all $s \in (0, s_*)$ but $y_* := x(s_*, \xi_*) \in \partial\mathcal{D}$.

The boundary $\partial\mathcal{D}$ near y_* can be defined by zero-level set of a function. More precisely, there exist a neighborhood \mathcal{U} of y_* and a function $G(\cdot) \in \mathcal{F}$ such that $\nabla G(x) \neq 0$ in \mathcal{U} , $G^{-1}(0) = \partial\mathcal{D} \cap \mathcal{U}$, $G(x) > 0$ in $\mathcal{U} \cap (\mathcal{M} \setminus \text{cl}(\mathcal{D}))$ and $G(x) < 0$ in $\mathcal{U} \cap \mathcal{D}$. Besides,

$$\nu(x) = \frac{1}{\|\nabla G(x)\|} \nabla G(x), \quad \langle II(x)\xi, \xi \rangle = \frac{1}{\|\nabla G(x)\|} \langle H_G(x)\xi, \xi \rangle, \quad \xi \in T_x \partial\mathcal{D}.$$

Now for sufficiently small $\delta > 0$ we have

$$g(s) := G(x(s, \xi_*)) < 0 \quad \forall s \in (s_* - \delta, s_*), \quad g(s_*) = G(x(s_*, \xi_*)) = 0. \quad (3.5)$$

Obviously

$$g'(s_*) := \frac{d}{ds} \Big|_{s=s_*} G(x(s, \xi_*)) \geq 0.$$

The case where $g'(s_*) > 0$ is impossible. In fact, in such a case there would exist $s' \in (s_*, 1)$ such that $G(x(s', \xi_*)) > 0$ and then

$$G(x(s', \xi_k)) > 0 \quad \Rightarrow \quad x(s', \xi_k) \notin \mathcal{D}$$

for all sufficiently large natural k . Thus, $g'(s_*) = 0$. Now observe that

$$\begin{aligned} g''(s_*) &= [\langle H_G(x)x', x' \rangle + \langle \nabla G(x), \nabla_{x'} x' \rangle]_{x=x(s_*, \xi_*)} = \\ &= \left[\langle H_G(x)x', x' \rangle + \frac{\|x'\|^2}{2} \langle \nabla G(x), U(x) \rangle \right]_{x=x(s_*, \xi_*)}. \end{aligned}$$

Since $\xi_* \neq 0$ then on account of (3.4) we have $x'(s_*, \xi_*) \neq 0$ and the condition (2.15) implies that $g''(s_*) > 0$. But then $g(\cdot)$ reaches its strict local minimum at the point $s = s_*$, and this produces a contradiction with (3.5).

Thus we have proved that \mathcal{X} is an open-close subset of open set \mathcal{D} . This implies that $\mathcal{X} = \mathcal{D}$. Now we can assert that for any $\{x_0, x_1\} \subset \mathcal{D}$ there exists $\xi(x_0, x_1) \in T_{x_0} \mathcal{M}$ such that $x(s, \xi(x_0, x_1)) \in \mathcal{D}$ for all $s \in [0, 1]$, and $x(1, \xi(x_0, x_1)) = x_1$.

By repeating the same arguments as above, we obtain the inequality

$$U(x_1) \geq U(x_0) - \|\nabla U(x_0)\| \|\xi\| + \frac{l_U}{2} \|\xi\|^2 \exp(U(x_0) - U(x(\theta, \xi)))$$

with $\xi = \xi(x_0, x_1)$ and some $\theta = \theta(x_0, x_1) \in (0, 1)$. Hence,

$$\frac{l_U}{2} e^{U_* - U^*} \|\xi\|^2 - C_U \|\xi\| - U^* + U_* \leq 0 \quad \Rightarrow \quad \|\xi\| \leq d.$$

Now let $\{x_0^*, x_1^*\} \subset \text{cl}(\mathcal{D})$. One can define sequences $\{x_i^k\} \subset \mathcal{D}$, $x_i^k \rightarrow x_i^*$, $k \rightarrow \infty$, $i \in \{0, 1\}$ such that $\xi(x_0^k, x_1^k) \rightarrow \xi^* \in T_{x_0^*} \mathcal{M}$, $\|\xi^*\| \leq d$. Then $x(1, \xi^*) = x_1^*$ and $x(s, \xi_*) \in \text{cl}(\mathcal{D})$ for all $s \in [0, 1]$. But actually the above arguments concerning the function $g(s)$ allow us to assert that there is no point $s_* \in (0, 1)$ such that $x(s_*, \xi_*) \in \partial\mathcal{D}$. \square

4. PROOFS OF THEOREMS

Proofs of Theorems 1 and 2. We proceed straight to the proof of Theorem 2.

Put $\bar{f}(x) := (2\pi)^{-k} \int_{\mathbb{T}^k} f(\varphi, x) d\varphi$ and observe that $\langle \nu(x), \bar{f}(x) \rangle > 0$ for all $x \in \partial\mathcal{D}$. For $s \in \mathbb{R}$ and $x \in \mathcal{M}$ let $t \mapsto X_s^t(x)$, $t \in I(s, x) \subset \mathbb{R}$, be the non-extendable solution of (1.2) satisfying the initial conditions

$$X_s^s(x) = x, \quad \dot{X}_s^s(x) = \epsilon \bar{f}(x) \quad \left(\dot{X}_s^t(x) := \frac{\partial}{\partial t} X_s^t(x) \right)$$

where $\epsilon > 0$ is small enough to ensure that $\|\epsilon \bar{f}(x)\| \leq z_+$. Hence, $\|\dot{X}_s^s(x)\|$ satisfies the same inequality as $\|\dot{x}(s)\|$ in Proposition 2. Let us show that there exists $x_{0,s} \in \mathcal{D}$ such that

$X_s^t(x_{0,s}) \in \mathcal{D}$ for all $t \geq s$. We shall exploit ideas of Wazewski topological principle. By reasoning ad absurdum, suppose that such a $x_{0,s}$ does not exist. Then

$$T(x) := \sup \{T > s : X_s^t(x) \in \mathcal{D} \forall t \in [s, T]\} < \infty \quad \forall x \in \mathcal{D}.$$

Obviously, $y(x) := X_s^{T(x)}(x) \in \partial\mathcal{D}$. By the same arguments as in the proof of Proposition 4, there exists a neighborhood \mathcal{U} of $y(x)$ and a function $G(\cdot) \in \mathcal{F}(\mathcal{M}) \mapsto \mathbb{R}$ such that $\nabla G(x) \neq 0$ in \mathcal{U} , $G^{-1}(0) = \partial\mathcal{D} \cap \mathcal{U}$, $G(x) > 0$ in $\mathcal{U} \cap (\mathcal{M} \setminus \text{cl}(\mathcal{D}))$, $G(x) < 0$ in $\mathcal{U} \cap \mathcal{D}$, and on account of $\lambda_{II}(x) > 0$ the Hesse form $H_G(x)$ is positive definite. There also exists $\delta(x) > 0$ such that

$$G(X_s^t(x)) < 0 \quad \forall t \in [T(x) - \delta(x), T(x)), \quad G(X_s^{T(x)}(x)) = 0. \quad (4.1)$$

Obviously that

$$\left. \frac{\partial}{\partial t} \right|_{t=T(x)} G(X_s^t(x)) = \left\langle \nabla G(y(x)), \dot{X}_s^{T(x)}(x)(x) \right\rangle \geq 0,$$

and since there holds the inequality (2.8), then

$$\begin{aligned} & \left. \frac{\partial^2}{\partial t^2} \right|_{t=T(x)} G(X_s^t(x)) = \\ & \left[\langle H_G(y) \xi, \xi \rangle + \langle \nabla G(y), f(t\omega, y) + P(t\omega, y)\xi \rangle \right] \Big|_{y=y(x), \xi=\dot{X}_s^{T(x)}(x)} = \\ & \|\nabla G(x)\| [\langle II(y)\xi, \xi \rangle + \langle \nu(y), f(t\omega, y) + P(t\omega, y)\xi \rangle] \Big|_{y=y(x), \xi=\dot{X}_s^{T(x)}(x)} \geq \\ & \|\nabla G(x)\| [\lambda_{II}(y) \|\xi\|^2 - \|P^*(t\omega, y)\nu(y)\| \|\xi\| + \langle \nu(y), f(t\omega, y) \rangle] \Big|_{y=y(x), \xi=\dot{X}_s^{T(x)}(x)} > 0. \end{aligned}$$

But now

$$\left. \frac{\partial}{\partial t} \right|_{t=T(x)} G(X_s^t(x)) > 0. \quad (4.2)$$

In fact, if the derivative in the right hand side of (4.2) were zero, then the function $t \mapsto G(X_s^t(x))$ would achieve a strict local minimum at $t = T(x)$. But this is impossible on account of (4.1).

The inequality (4.2) together with the inverse function theorem implies that $T(\cdot) : \mathcal{D} \mapsto \mathbb{R}$ is smooth. Let now $y \in \mathcal{U} \cap \partial\mathcal{D}$. Then

$$G(X_s^s(y)) = 0, \quad \left. \frac{\partial}{\partial t} \right|_{t=s} G(X_s^t(y)) = \|\nabla G(y)\| \langle \nu(y), \bar{f}(y) \rangle > 0.$$

Hence, the function $T(\cdot)$ is also smooth in a small neighborhood of y and $T(y) = s$. It follows from the above that the mapping $\rho(\cdot, \cdot) : \text{cl}(\mathcal{D}) \times [0, 1] \mapsto \text{cl}(\mathcal{D})$ defined by the formula

$$\text{cl}(\mathcal{D}) \times [0, 1] \ni (x, u) \mapsto X_s^{uT(x) + (1-u)s}(x) \in \text{cl}(\mathcal{D})$$

is a deformation retraction of $\text{cl}(\mathcal{D})$ onto $\partial\mathcal{D}$. We reach a contradiction, since the boundary of compact manifold $\text{cl}(\mathcal{D})$ cannot be a retract of $\text{cl}(\mathcal{D})$.

Thus we have proved that $x_{0,s}$ does exist. Now from Propositions 1 and 2 it follows that the solution defined by $x_s(t) := X_s^t(x_{0,s})$ satisfies

$$x_s(\cdot) : [s, \infty) \mapsto \mathcal{D}, \quad \sup_{t \geq s} \|\dot{x}_s(t)\| \leq z_*. \quad (4.3)$$

Consider the sequence $\{x_{-i}(\cdot) : [-i, \infty) \mapsto \mathcal{D}\}_{i \in \mathbb{N}}$. Obviously that $x_{-i}(\cdot)$ is the solution of (1.2) satisfying the initial conditions

$$x|_{t=0} = x_{-i}(0), \quad \dot{x}|_{t=0} = \dot{x}_{-i}(0).$$

From (4.3) it follows that there exists a sub-sequence $i_j \rightarrow \infty$, $j \rightarrow \infty$, such that the sequence $\{(x_{-i_j}(0), \dot{x}_{-i_j}(0))\}_{j \in \mathbb{N}}$ converges to a point (x_0, ξ_0) such that $x_0 \in \text{cl}(\mathcal{D})$, $\xi_0 \in T_{x_0}\mathcal{M}$,

$\|\xi_0\| \leq C_*$. Now it is not hard to see that the non-extendable solution $x_*(\cdot)$ of (1.2) satisfying the initial conditions

$$x_*(0) = x_0, \quad \dot{x}_*(0) = \xi_0$$

is defined on the whole real line and satisfies the conditions

$$x_*(\cdot) : \mathbb{R} \mapsto \text{cl}(\mathcal{D}), \quad \sup_{t \in \mathbb{R}} \|\dot{x}(t)\| \leq z_*.$$

In fact, if this were not true, then there would exist $t' \in \mathbb{R}$ such that either $x_*(t') \in \mathcal{M} \setminus \text{cl}(\mathcal{D})$ or $\|\dot{x}_*(t')\| > z_*$, and then, respectively, either $x_{i_j}(t') \notin \mathcal{D}$ or $\|\dot{x}_{i_j}(t')\| > z_*$ for all sufficiently large j . This is impossible on account of (4.3).

The same arguments as above (see also [13]) allows us to show that $x_*(\cdot) : \mathbb{R} \mapsto \mathcal{D}$. In fact, if there were a moment t_0 such that $x_*(t_0) \in \partial\mathcal{D}$, then

$$G(x_*(t_0)) = 0, \quad \frac{d}{dt}\bigg|_{t=t_0} G(x_*(t)) = 0, \quad \frac{d^2}{dt^2}\bigg|_{t=t_0} G(x_*(t)) > 0.$$

This implies that the function $G(x_*(\cdot))$ achieves a strict local minimum at $t = t_0$, and hence $\{t \in \mathbb{R} : G(x_*(t)) > 0\} \neq \emptyset$. We reach a contradiction. Q.E.D.

Proofs of Theorems 3 and 4. We start with the proof of Theorem 4. Let $x(\cdot) : \mathbb{R} \mapsto \mathcal{D}$ be a solution of System (1.2) such that $\sup_{t \in \mathbb{R}} \|\dot{x}(t)\| := Z < \infty$. Put $\tau(t) = \dot{x}(t)$, introduce the non-degenerate quadratic form

$$Q(\eta, \zeta; t) = \langle \eta, \zeta \rangle + \frac{\|\eta\|^2}{2} \langle \nabla U(x(t)), \tau(t) \rangle, \quad \eta, \zeta \in T_{x(t)}\mathcal{M}$$

and find its derivative along solution of the linear variational system with respect to $x(t)$:

$$\begin{aligned} \dot{Q}(\eta, \zeta; t) &= \|\dot{\zeta}\|^2 + \langle \eta, \nabla f \eta \rangle - \langle R(\eta, \tau) \tau, \eta \rangle + \langle \nabla P(\eta, \tau), \eta \rangle + \langle P \zeta, \eta \rangle + \\ &\quad \langle \eta, \zeta \rangle \langle \nabla U, \tau \rangle + \frac{\|\eta\|^2}{2} (\langle H_U \tau, \tau \rangle + \langle \nabla U, f + P \tau \rangle) \end{aligned}$$

For the sake of simplifying the calculations, here and below we do not show explicitly the arguments $\varphi = t\omega$, $x = x(t)$. Taking into account (2.11), (2.12), (2.13), (2.13) we have

$$\begin{aligned} \dot{Q}(\eta, \zeta; t) &\geq \\ &\geq \|\dot{\zeta}\|^2 - \|\eta\| \|\dot{\zeta}\| (|\langle \nabla U, \tau \rangle| + M_P) + \|\eta\|^2 \left[\frac{\langle H_U \tau, \tau \rangle}{2} - K \|\tau\|^2 - L_P \|\tau\| \right] + \\ &\quad \|\eta\|^2 \left[\lambda_f + \frac{1}{2} \langle \nabla U, f \rangle - \frac{M_{PU}}{2} \|\tau\| \right] \geq \\ &\quad \left[\|\dot{\zeta}\| - \frac{1}{2} \|\eta\| |\langle \nabla U, \tau \rangle| \right]^2 + \frac{\|\eta\|^2 \|\tau\|^2}{2} [\mu_U - 2K] + \\ &\quad + \|\eta\|^2 \left[\lambda_f + \frac{\langle \nabla U, f \rangle}{2} \right]_{x=x(t)} - M_P \|\dot{\zeta}\| \|\eta\| - \left(L_P + \frac{M_{PU}}{2} \right) Z \|\eta\|^2, \end{aligned}$$

Since there holds the inequality (2.14) and $|\langle \nabla U, \tau \rangle| \leq M_U Z$, then there exists a constant $\alpha_1 > 0$ such that

$$\dot{Q}(\eta, \zeta; t) \geq \alpha_1 (\|\dot{\zeta}\|^2 + \|\eta\|^2).$$

Let $\Theta_s^t : T_{x_*(s)}\mathcal{M} \mapsto T_{x_*(t)}\mathcal{M}$ be the cocycle of parallel shift along the mapping $x_*(\cdot)$ from point $x_*(s)$ to point $x_*(t)$. For any $\eta \in T_{x_*(s)}\mathcal{M}$, there holds

$$\nabla_{\tau(t)} \Theta_s^t \eta = 0 \quad \forall t \in \mathbb{R}, \quad \Theta_s^s = \text{Id}, \quad \Theta_s^t \Theta_r^s = \Theta_r^t.$$

After the change of variables

$$\eta = \Theta_0^t y, \quad \zeta = \Theta_0^t z, \quad y, z \in T_{x_*(0)}\mathcal{M}$$

the system in variations takes the form

$$\dot{y} = z, \quad \dot{z} = A(t)y \quad (4.4)$$

where

$$A(t) := \Theta_t^0 \left[\nabla f(t\omega, x) \Theta_0^t y - R(\Theta_0^t y, \tau) \tau \right]_{x=x_*(t), \tau=\tau_*(t)} + \\ \left[\nabla P(t\omega, x) (\Theta_0^t y, \tau) + P(t\omega, x) \Theta_0^t z \right]_{x=x_*(t), \tau=\tau_*(t)},$$

$\tau_*(t) := \dot{x}_*(t)$. Since $\langle \Theta_0^t y, \Theta_0^t z \rangle = \langle y, z \rangle$, it follows from the above that the derivative of quadratic form

$$Q_0(y, z; t) := \langle y, z \rangle + \frac{\|y\|^2}{2} \langle \nabla U(x_*(t)), \tau(t) \rangle$$

along solutions of System (4.4) is positive definite. It is known that the existence of non-degenerate quadratic form with the above property implies that System (4.4) is exponentially dichotomic [26]. Q.E.D.

The proof of Theorem 3 is obviously follows from the above one by letting $P(\varphi, x) = 0$.

Proofs of Theorems 5 and 6. Let us proceed to the proof of Theorem 6. Theorem 5 will immediately follow from Theorem 6. By Theorem 2 the domain \mathcal{D} contains a solution $x_*(\cdot) : \mathbb{R} \mapsto \mathcal{D}$ of System (2.11) such that $\sup_{t \in \mathbb{R}} \|\dot{x}_*(t)\| \leq z_*$. Let us show that the solution with the above properties is unique. Suppose that there exist two solutions $x_i(\cdot) : \mathbb{R} \mapsto \mathcal{D}$ of System (1.2) such that $\sup_{t \in \mathbb{R}} \|\dot{x}_i(t)\| \leq z_*$, $i \in \{1, 2\}$. By Propositions 3 and 4, with the help of implicit function theorem and a continuation procedure one can construct a smooth vector field $\xi(\cdot) : \mathbb{R} \mapsto T\mathcal{M}$ along $x_1(\cdot)$ such that

$$\xi(t) \in T_{x_1(t)}\mathcal{M}, \quad \|\xi(t)\| \leq d, \quad \frac{\partial}{\partial s} \Big|_{s=0} x(s, \xi(t)) = \xi(t), \\ x(0, \xi(t)) = x_1(t), \quad x(1, \xi(t)) = x_2(t).$$

Introduce the smooth mapping $\chi(\cdot, \cdot) : [0, 1] \times \mathbb{R} \mapsto \mathcal{D}$ by the equality $\chi(s, t) := x(s, \xi(t))$ and define the tangent vector fields $\chi'(\cdot, \cdot)$, $\dot{\chi}(\cdot, \cdot)$ along this mapping as

$$\chi'(s, t) := \frac{\partial}{\partial s} x(s, \xi(t)), \quad \dot{\chi}(s, t) := \frac{\partial}{\partial t} x(s, \xi(t)).$$

Define also the function

$$S(t) := \langle \chi', \dot{\chi} \rangle \Big|_{s=0}^{s=1} \equiv \langle \chi'(1, t), \dot{x}_2(t) \rangle - \langle \xi(t), \dot{x}_1(t) \rangle \quad (4.5)$$

and calculate its derivative:

$$\dot{S}(t) = \left[\frac{d}{dt} \langle \chi', \dot{\chi} \rangle \right] \Big|_{s=0}^{s=1} = [\langle \nabla_{\dot{\chi}} \chi', \dot{\chi} \rangle + \langle \chi', \nabla_{\dot{\chi}} \dot{\chi} \rangle] \Big|_{s=0}^{s=1} = \\ [\langle \nabla_{\chi'} \dot{\chi}, \dot{\chi} \rangle + \langle \chi', \nabla_{\dot{\chi}} \dot{\chi} \rangle] \Big|_{s=0}^{s=1} = \\ \left[\frac{\partial}{\partial s} \frac{\|\dot{\chi}\|^2}{2} \right]_{s=0}^{s=1} + \langle \chi', f(t\omega, \chi) + P(t\omega, \chi) \dot{\chi} \rangle \Big|_{s=0}^{s=1} = \\ \int_0^1 \left[\frac{\partial^2}{\partial s^2} \frac{\|\dot{\chi}\|^2}{2} + \frac{\partial}{\partial s} \langle \chi', f(t\omega, \chi) + P(t\omega, \chi) \dot{\chi} \rangle \right] ds.$$

Using the equalities

$$\nabla_{\dot{\chi}} \chi' = \nabla_{\chi'} \dot{\chi}, \quad \nabla_{\chi'}^2 \dot{\chi} = \nabla_{\chi'} \nabla_{\dot{\chi}} \chi' = \nabla_{\dot{\chi}} \nabla_{\chi'} \chi' - R(\dot{\chi}, \chi') \chi', \\ \nabla_{\chi'} \chi' = \|\chi'\|^2 \nabla U(\chi)/2,$$

we obtain

$$\begin{aligned}
\frac{\partial^2}{\partial s^2} \frac{\|\dot{\chi}\|^2}{2} &= \|\nabla_{\chi'} \dot{\chi}\|^2 + \langle \nabla_{\dot{\chi}} \nabla_{\chi'} \chi', \dot{\chi} \rangle - \langle R(\dot{\chi}, \chi') \chi', \dot{\chi} \rangle = \\
&\|\nabla_{\dot{\chi}} \chi'\|^2 + \langle \nabla_{\dot{\chi}} \chi', \chi' \rangle \langle \nabla U(\chi), \dot{\chi} \rangle + \frac{\|\chi'\|^2}{2} \langle H_U(\chi) \dot{\chi}, \dot{\chi} \rangle - \\
&\quad K(\chi) \left[\|\chi'\|^2 \|\dot{\chi}\|^2 - \langle \chi', \dot{\chi} \rangle^2 \right] \geq \\
&\|\nabla_{\dot{\chi}} \chi'\|^2 - \|\nabla_{\dot{\chi}} \chi'\| \|\chi'\| |\langle \nabla U(\chi), \dot{\chi} \rangle| + \|\chi'\|^2 \left(\frac{1}{2} \langle H_U(\chi) \dot{\chi}, \dot{\chi} \rangle - K(\chi) \|\dot{\chi}\|^2 \right) = \\
&\left[\|\nabla_{\dot{\chi}} \chi'\| - \frac{|\langle \nabla U(\chi), \dot{\chi} \rangle|}{2} \right]^2 + \frac{\|\chi'\|^2}{2} [\mu_U(\chi) - 2K(\chi)] \geq 0; \\
\frac{\partial}{\partial s} \langle f(\varphi, \chi), \chi' \rangle &= \langle \nabla f(\varphi, \chi) \chi', \chi' \rangle + \frac{\|\chi'\|^2}{2} \langle f(\varphi, \chi), \nabla U(\chi) \rangle \geq \\
&\|\chi'\|^2 \left[\lambda_f(\varphi, \chi) + \frac{1}{2} \langle f(\varphi, \chi), \nabla U(\chi) \rangle \right].
\end{aligned}$$

Since $\frac{\partial^2}{\partial s^2} \|\dot{\chi}(s, t)\|^2 \geq 0$, then

$$\|\dot{\chi}(s, t)\|^2 \leq \frac{1}{2} [\|\dot{\chi}(0, t)\|^2 + \|\dot{\chi}(1, t)\|^2] = \frac{1}{2} [\|\dot{x}_1(t)\|^2 + \|\dot{x}_2(t)\|^2] \leq z_*^2.$$

Now

$$\begin{aligned}
\left| \frac{\partial}{\partial s} \langle P(\varphi, x) \dot{\chi}, \chi' \rangle \right| &\leq |\langle \nabla_{\chi'} [P(\varphi, x) \dot{\chi}], \chi' \rangle| + |\langle P(\varphi, x) \dot{\chi}, \nabla_{\chi'} \chi' \rangle| \leq \\
&|\langle \nabla P(\varphi, x)(\chi', \dot{\chi}), \chi' \rangle| + |\langle P(\varphi, x) \nabla_{\chi'} \dot{\chi}, \chi' \rangle| + |\langle P(\varphi, x) \dot{\chi}, \nabla_{\chi'} \chi' \rangle| \leq \\
&L_P(\varphi, x) z_* \|\chi'\|^2 + M_P(\varphi, x) \|\nabla_{\dot{\chi}} \chi'\| \|\chi'\| + \frac{M_{PU}(\varphi, x) z_*}{2} \|\chi'\|^2.
\end{aligned}$$

Just like in the proof of Theorem (4), it is not hard to show that the above inequalities together with condition (2.16) imply that there exists $\alpha_2 > 0$ such that there holds the inequality

$$\dot{S}(t) \geq \alpha_2 \int_0^1 \left(\|\nabla_{\dot{\chi}} \chi'\|^2 + \|\chi'\|^2 \right) ds.$$

By Proposition 4 $\sup_{t \in \mathbb{R}} \|\xi(t)\| \leq d$, and on account of (3.4) and (3.3) we have

$$\|\xi(t)\|^2 e^{U_* - U^*} \leq \|\chi'(s, t)\|^2 = \|\xi(t)\|^2 \exp(U(x_1(t)) - U(\chi(s, t))) \leq \|\xi(t)\|^2 e^{U^* - U_*}. \quad (4.6)$$

Thus we obtain the inequality $\dot{S}(t) \geq \alpha_2 e^{U_* - U^*} \|\xi(t)\|^2$ which together with boundedness of $\|\xi(t)\|$ and $|S(t)|$ yields that

$$-\infty < S(-\infty) < S(0) < S(+\infty) < \infty.$$

Now it turns out that either $l_+ := \liminf_{t \rightarrow +\infty} \|\xi(t)\| > 0$ or $l_- := \liminf_{t \rightarrow -\infty} \|\xi(t)\| > 0$. In fact, if $l_- = l_+ = 0$ then (4.5) and (4.6) implies that $S(-\infty) = S(+\infty) = 0$ and we reach the contradiction. But if $l_+ > 0$, then $S(+\infty) = +\infty$, and if $l_- > 0$ then $S(-\infty) = -\infty$. Both these cases produce the contradiction. Hence, we have proved the announced uniqueness.

Obviously, the above reasoning is valid also for any system of the form

$$\nabla_{\dot{x}} \dot{x} = f(t\omega + \varphi, x) + P(t\omega + \varphi, x) \dot{x} \quad \forall \varphi \in \mathbb{T}^k.$$

Hence, for any $\varphi \in \mathbb{T}^k$ there exists the solution $x_*(\cdot, \varphi) : \mathbb{R} \mapsto \mathcal{D}$ which generates the single valued mapping $x_*(\cdot, \cdot) : \mathbb{R} \times \mathbb{T}^k \mapsto \mathcal{D}$ such that $\sup_{(t, \varphi) \in \mathbb{R} \times \mathbb{T}^k} \|\dot{x}_*(t, \varphi)\| \leq z_*$. But then

$$x_*(t + s, \varphi) = x_*(s, t\omega + \varphi) \quad \forall \{t, s\} \subset \mathbb{R}, \varphi \in \mathbb{T}^k.$$

If we put here $s = 0$ and define the mapping $h(\cdot) := x_*(0, \cdot) : \mathbb{T}^k \mapsto \mathcal{D}$ then we obtain

$$x_*(t, \varphi) = h(t\omega + \varphi).$$

To show that $x_*(t, \varphi)$ is quasiperiodic, let us prove that $h(\cdot)$ is continuous. Suppose that the opposite is true. Then there exists $\varphi_* \in \mathbb{T}^k$ and a sequence $\{\varphi_i\} \subset \mathbb{T}^k$ converging to φ_* such that

$$h(\varphi_i) \rightarrow \tilde{x}_0 \in \text{cl}(\mathcal{D}), \quad \dot{x}_*(0, \varphi_i) \rightarrow \tilde{\xi}_0, \quad \|\tilde{\xi}_0\| \leq z_*, \quad \tilde{x}_0 \neq x_*(0, \varphi_*).$$

Consider the non-extendable solution $\tilde{x}(\cdot) : I \mapsto \mathcal{M}$ of the initial-value problem

$$\nabla_{\dot{x}} \dot{x} = f(t\omega + \varphi_*, x) + P(t\omega + \varphi_*, x)\dot{x}, \quad x(0) = \tilde{x}_0, \quad \dot{x}(0) = \tilde{\xi}_0.$$

Since each system $\nabla_{\dot{x}} \dot{x} = f(t\omega + \varphi_i, x) + P(t\omega + \varphi_i, x)\dot{x}$ is equivalent to the first order system

$$\dot{x} = \eta, \quad \nabla_{\dot{x}} \eta = f(t\omega + \varphi_i, x) + P(t\omega + \varphi_i, x)\dot{x}$$

and $\{f(\varphi + \varphi_i, x) + P(\varphi + \varphi_i, x)\dot{x}\}$ converges to $f(\varphi + \varphi_*, x) + P(\varphi + \varphi_*, x)\dot{x}$ uniformly with respect to $\varphi \in \mathbb{T}^k$, $x \in \text{cl}(\mathcal{D})$, $\dot{x} \in T_x \mathcal{M}$, and $\|\dot{x}\| \leq z_*$, then for any closed segment $J \subset I$ the sequence $\{(x_*(t, \varphi_i), \dot{x}_*(t, \varphi_i))\}$ converges to $(\tilde{x}(t), \frac{d}{dt}\tilde{x}(t))$ uniformly with respect to $t \in J$. This yields that $\tilde{x}(\cdot) : I \mapsto \text{cl}(\mathcal{D})$, $\sup_{t \in I} \|\frac{d}{dt}\tilde{x}(t)\| \leq z_*$ and hence $I = \mathbb{R}$. The same arguments as in the proof of Theorem 2 allow us to show that $\tilde{x}(\cdot) : \mathbb{R} \mapsto \mathcal{D}$. Thus we reach the contradiction with uniqueness of solution taking values in \mathcal{D} and possessing the derivative of norm bounded by z_* .

5. QUASIPERIODIC MOTION OF CHARGED PARTICLE ON UNIT SPHERE

Let $\mathbb{E}^3 = (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ be the 3-dimensional Euclidean space endowed with a scalar product $\langle \cdot, \cdot \rangle$ and cross-product $\cdot \times \cdot$. Consider a charged particle of unit mass which is constrained to move on the surface of the sphere $\mathbb{S}^2 := \{\mathbf{x} \in \mathbb{E}^3 : \|\mathbf{x}\|^2 = 1\}$ by the applied force Φ represented in the form

$$\Phi(t\omega, \mathbf{x}, \dot{\mathbf{x}}) = -\frac{\mathbf{x} - \mathbf{a}}{\|\mathbf{x} - \mathbf{a}\|^3} + \mathbf{E}(t\omega) + \dot{\mathbf{x}} \times \mathbf{B}(t\omega) - \kappa \dot{\mathbf{x}}.$$

Here $\mathbf{a} \in \mathbb{E}^3$ is a vector of norm $a := \|\mathbf{a}\|$; b, κ are positive parameters; $\mathbf{E}(\cdot) : \mathbb{T}^k \mapsto \mathbb{E}^3$ and $\mathbf{B}(\cdot) : \mathbb{T}^k \mapsto \mathbb{E}^3$ are smooth mappings; $\omega \in \mathbb{R}^k$ is a frequency vector. The force Φ can be naturally interpreted as the superposition of three forces: the Coulomb force caused by a charge placed at point \mathbf{a} ; the Lorentz force caused by the electric field \mathbf{E} and the magnetic field \mathbf{B} ; the damping force $-\kappa \dot{\mathbf{x}}$.

Subtracting from $\Phi(t\omega, \mathbf{x}, \dot{\mathbf{x}})$ its normal component and introducing unit vector $\mathbf{k} := -\mathbf{a}/a$, we find that in the case under consideration the forces affecting the motion of the constrained particle are

$$\begin{aligned} \mathbf{f}(t\omega, \mathbf{x}) &= -\frac{\mathbf{x} + a\mathbf{k}}{\|\mathbf{x} + a\mathbf{k}\|^3} + \mathbf{E}(t\omega) + \left\langle \frac{\mathbf{x} + a\mathbf{k}}{\|\mathbf{x} + a\mathbf{k}\|^3} - \mathbf{E}(t\omega), \mathbf{x} \right\rangle \mathbf{x}, \\ P(t\omega, \mathbf{x})\dot{\mathbf{x}} &= \dot{\mathbf{x}} \times \mathbf{B}(t\omega) - \langle \dot{\mathbf{x}} \times \mathbf{B}(t\omega), \mathbf{x} \rangle \mathbf{x} - \kappa \dot{\mathbf{x}}. \end{aligned}$$

Recall that if $\mathbf{v}(\cdot) : \mathbb{S}^2 \mapsto \mathbb{E}^3$ is a smooth tangent vector field on \mathbb{S}^2 , i.e. $\langle \mathbf{v}(\mathbf{x}), \mathbf{x} \rangle = 0$ for any $\mathbf{x} \in \mathbb{S}^2$, then for any $\mathbf{h} \in T_{\mathbf{x}} \mathbb{S}^2$ we have

$$\nabla_{\mathbf{h}} \mathbf{v}(\mathbf{x}) = \mathbf{v}'(\mathbf{x})\mathbf{h} - \langle \mathbf{v}'(\mathbf{x})\mathbf{h}, \mathbf{x} \rangle \mathbf{x}.$$

First consider the case where the influence of magnetic field and the damping force can be neglected.

Theorem 7. *Let $\mathbf{B}(\varphi) \equiv 0$, $\kappa = 0$. If there holds the inequality*

$$\frac{a}{(1+a)^3} - \langle \mathbf{E}(\varphi), \mathbf{k} \rangle > 0 \quad \forall \varphi \in \mathbb{T}^k \quad (5.1)$$

and there exists a point $\varphi_0 \in \mathbb{T}^k$ such that $\mathbf{E}(\varphi_0) \nparallel \mathbf{k}$, then the system of charged particle on \mathbb{S}^1 has a unique ω -quasiperiodic solution located in the hemisphere $\{\mathbf{x} \in \mathbb{S}^1 : 0 < \langle \mathbf{x}, \mathbf{k} \rangle \leq 1\}$. This solution is hyperbolic.

Proof. Let a unit tangent vector $\mathbf{e} \in T_{\mathbf{x}}\mathbb{S}^1$ be taken at will. Then in view of $\langle \mathbf{x}, \mathbf{e} \rangle = 0$ we have

$$\begin{aligned} \langle \nabla_{\mathbf{e}} \mathbf{f}(\varphi, \mathbf{x}), \mathbf{e} \rangle &= \langle \mathbf{f}'_{\mathbf{x}}(\varphi, \mathbf{x}) \mathbf{e}, \mathbf{e} \rangle = \\ &= -\frac{1}{\|\mathbf{x} + a\mathbf{k}\|^3} + \frac{3a^2 \langle \mathbf{k}, \mathbf{e} \rangle^2}{\|\mathbf{x} + a\mathbf{k}\|^5} + \left\langle \frac{\mathbf{x} + a\mathbf{k}}{\|\mathbf{x} + a\mathbf{k}\|^3} - \mathbf{E}(\varphi), \mathbf{x} \right\rangle = \\ &= \frac{a \langle \mathbf{k}, \mathbf{x} \rangle}{\|\mathbf{x} + a\mathbf{k}\|^3} + \frac{3a^2 \langle \mathbf{k}, \mathbf{e} \rangle^2}{\|\mathbf{x} + a\mathbf{k}\|^5} - \langle \mathbf{E}(\varphi), \mathbf{x} \rangle. \end{aligned}$$

It is not hard to see that

$$\lambda_{\mathbf{f}}(\varphi, \mathbf{x}) = \frac{a \langle \mathbf{k}, \mathbf{x} \rangle}{\|\mathbf{x} + a\mathbf{k}\|^3} - \langle \mathbf{E}(\varphi), \mathbf{x} \rangle.$$

In particular,

$$\lambda_{\mathbf{f}}(\varphi, \mathbf{k}) = \frac{a}{(1+a)^3} - \langle \mathbf{E}(\varphi), \mathbf{k} \rangle > 0.$$

Now, in order to apply Theorem 5, we are going to find the appropriate domain \mathcal{D} and function $U(\cdot)$. Observe that for a function $U(\cdot) : \mathbb{S}^2 \mapsto \mathbb{R}$ such that $\nabla U(\mathbf{k}) = 0$, the inequality(2.9) holds true at least near \mathbf{k} . For this reason, we define the domain

$$\mathcal{D} := \{\mathbf{x} \in \mathbb{S}^2 : \rho < \langle \mathbf{k}, \mathbf{x} \rangle \leq 1\}$$

where $\rho \in (0, 1)$ will be determined later.

Set $u(\mathbf{x}) := -\langle \mathbf{k}, \mathbf{x} \rangle$. To satisfy the conditions of Theorem 5 we seek the function $U(\cdot)$ in the form $U(\mathbf{x}) = y \circ u(\mathbf{x})$. For $\mathbf{e} \in T_{\mathbf{x}}\mathbb{S}^2$, $\|\mathbf{e}\| = 1$, we have

$$\nabla_{\mathbf{e}} \nabla U(\mathbf{x}) = [y''(u) \langle \nabla u, \mathbf{e} \rangle \nabla u + y'(u) \nabla_{\mathbf{e}} \nabla u]_{u=u(\mathbf{x})}.$$

On account that

$$\nabla u(\mathbf{x}) = -\mathbf{k} + \langle \mathbf{k}, \mathbf{x} \rangle \mathbf{x} = -u(\mathbf{x})\mathbf{x} - \mathbf{k}, \quad \nabla_{\mathbf{e}} \nabla u(\mathbf{x}) = -u(\mathbf{x})\mathbf{e}, \quad (5.2)$$

we obtain

$$\begin{aligned} \langle H_U(\mathbf{x}) \mathbf{e}, \mathbf{e} \rangle &= [y''(u) \langle \mathbf{k}, \mathbf{e} \rangle^2 + y'(u) \langle \mathbf{k}, \mathbf{x} \rangle]_{u=u(\mathbf{x})} = [\langle \mathbf{k}, \mathbf{e} \rangle^2 y''(u) - u y'(u)]_{u=u(\mathbf{x})}, \\ \langle \nabla U(\mathbf{x}), \mathbf{e} \rangle^2 &= y'^2(u(\mathbf{x})) \langle \nabla u(\mathbf{x}), \mathbf{e} \rangle^2 = \langle \mathbf{k}, \mathbf{e} \rangle^2 y'^2(u(\mathbf{x})). \end{aligned}$$

As is well known, $K(\mathbf{x}) = 1$ for $\mathcal{M} = \mathbb{S}^2$, and in our case the inequality $\mu_U(x) \geq 2K(x)$ of Hypothesis **H3** takes the form

$$[\langle \mathbf{k}, \mathbf{e} \rangle^2 (y''(u) - y'^2(u)/2) - u y'(u)]_{u=u(\mathbf{x})} \geq 2 \quad \forall u \in [-1, -\rho].$$

This inequality obviously turns into equality if we put $y = -\ln u^2$. Hence, it is naturally to define

$$U(\mathbf{x}) := -\ln u^2(\mathbf{x}).$$

Let us verify Hypothesis **H1**. Under the above choice of $U(\cdot)$, we get

$$\nabla U(\mathbf{x}) = -\frac{2\nabla u(\mathbf{x})}{u(\mathbf{x})}, \quad \langle H_U(\mathbf{x}) \mathbf{e}, \mathbf{e} \rangle = \frac{2 \langle \mathbf{k}, \mathbf{e} \rangle^2}{u^2(\mathbf{x})} + 2 \quad \Rightarrow \quad \lambda_U(\mathbf{x}) = 2. \quad (5.3)$$

Since the third addendum in expression for $\mathbf{f}(\varphi, \mathbf{x})$ is orthogonal to \mathbb{S}^2 , then on account of (5.2)

$$\begin{aligned} \langle \nabla U(\mathbf{x}), \mathbf{f}(\varphi, \mathbf{x}) \rangle &= \frac{2}{u(\mathbf{x})} \left\langle \nabla u(\mathbf{x}), \frac{\mathbf{x} + a\mathbf{k}}{\|\mathbf{x} + a\mathbf{k}\|^3} - \mathbf{E}(\varphi) \right\rangle = \\ &= -\frac{2}{u(\mathbf{x})} \left[\frac{a(1 - u^2(\mathbf{x}))}{\|\mathbf{x} + a\mathbf{k}\|^3} - \langle \mathbf{E}(\varphi), \mathbf{k} + u(\mathbf{x})\mathbf{x} \rangle \right]. \end{aligned} \quad (5.4)$$

We have to show that this function has negative minimum in $\mathbb{T}^k \times \text{cl}(\mathcal{D})$, or, what is the same, the parameter q (see (2.5)) is correctly defined, i.e. actually is positive. Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the standard right-oriented orthonormal basis in \mathbb{E}^3 . Denote by $E_{\mathbf{i}}(\varphi), E_{\mathbf{j}}(\varphi), E_{\mathbf{k}}(\varphi)$ the projections of $\mathbf{E}(\varphi)$ onto $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively. It turns out that for fixed $\varphi \in \mathbb{T}^k$ and $s \in [\rho, 1]$ the conditional maximum

$$M(s, \varphi) := \max \left\{ -\frac{\langle \nabla U(x), \mathbf{f}(\varphi, \mathbf{x}) \rangle}{\lambda_U(\mathbf{x})} : \langle \mathbf{k}, \mathbf{x} \rangle = s \right\}$$

is attained at point $\mathbf{x} \in \mathbb{S}^2$ such that

$$\langle \mathbf{i}, \mathbf{x} \rangle = -\frac{\sqrt{1-s^2}E_{\mathbf{i}}(\varphi)}{\sqrt{E_{\mathbf{i}}^2(\varphi) + E_{\mathbf{j}}^2(\varphi)}}, \quad \langle \mathbf{j}, \mathbf{x} \rangle = -\frac{\sqrt{1-s^2}E_{\mathbf{j}}(\varphi)}{\sqrt{E_{\mathbf{i}}^2(\varphi) + E_{\mathbf{j}}^2(\varphi)}}, \quad \langle \mathbf{k}, \mathbf{x} \rangle = s.$$

Hence,

$$M(s, \varphi) = -\frac{a(1-s^2)}{s(1+2sa+a^2)^{3/2}} + \frac{(1-s^2)E_{\mathbf{k}}(\varphi)}{s} + \sqrt{1-s^2}\sqrt{E_{\mathbf{i}}^2(\varphi) + E_{\mathbf{j}}^2(\varphi)}.$$

Since there exists a point $\varphi_0 \in \mathbb{T}^k$ such that $\mathbf{E}(\varphi_0) \not\parallel \mathbf{k}$, then $M(s, \varphi_0) > 0$ if s is sufficiently close to 1. Hence we show that

$$q^2 = \max \{ M(s, \varphi) : \varphi \in \mathbb{T}^k, s \in [\rho, 1] \} > 0$$

and that Hypothesis **H1** is satisfied.

Let us verify Hypothesis **H2**. Since $\partial\mathcal{D} := \{\mathbf{x} \in \mathbb{S}^2 : u(\mathbf{x}) = -\rho\}$, then on account of (5.2) the outward unit normal at $\mathbf{x} \in \partial\mathcal{D}$ is

$$\mathbf{n}(\mathbf{x}) = \frac{\nabla u(\mathbf{x})}{\|\nabla u(\mathbf{x})\|} = \frac{\nabla u(\mathbf{x})}{\sqrt{1-u^2(\mathbf{x})}} = \frac{\rho}{2\sqrt{1-\rho^2}} \nabla U(\mathbf{x}), \quad \mathbf{x} \in \partial\mathcal{D}.$$

Now from (5.2), (5.3) it follows that

$$\lambda_{II}(\mathbf{x}) = \frac{\langle \nabla_{\mathbf{e}} \nabla u(\mathbf{x}), \mathbf{e} \rangle}{\|\nabla u(\mathbf{x})\|} = \frac{\rho}{\sqrt{1-\rho^2}} > 0 \quad \forall \mathbf{x} \in \partial\mathcal{D}.$$

Observe that $\mathbf{n}(\mathbf{x}) = \frac{\rho}{2\sqrt{1-\rho^2}} \nabla U(\mathbf{x})$. Now (5.4) yields

$$\begin{aligned} \min \{ \langle \mathbf{n}(\mathbf{x}), \mathbf{f}(\varphi, \mathbf{x}) \rangle : \mathbf{x} \in \partial\mathcal{D} \} &= -\frac{\rho}{\sqrt{1-\rho^2}} M(\rho, \varphi) = \\ &= \sqrt{1-\rho^2} \left[\frac{a}{(1+2\rho a+a^2)^{3/2}} - E_{\mathbf{k}}(\varphi) \right] - \rho \sqrt{E_{\mathbf{i}}^2(\varphi) + E_{\mathbf{j}}^2(\varphi)}, \end{aligned}$$

$$\lambda_{II}(\mathbf{x}) \langle \mathbf{n}(\mathbf{x}), \mathbf{f}(\varphi, \mathbf{x}) \rangle \geq \frac{\rho}{2} \left[\frac{a}{(1+2\rho a+a^2)^{3/2}} - E_{\mathbf{k}}(\varphi) \right] - \frac{\rho^2}{2} \sqrt{\frac{E_{\mathbf{i}}^2(\varphi) + E_{\mathbf{j}}^2(\varphi)}{1-\rho^2}}.$$

The condition (5.1) implies that here the right-hand sides of both inequalities are positive once ρ is sufficiently small. Besides,

$$\langle \nabla U(\mathbf{x}), \mathbf{n}(\mathbf{x}) \rangle = -\frac{2\|\nabla u(\mathbf{x})\|^2}{u(\mathbf{x})\sqrt{1-u^2(\mathbf{x})}} = \frac{2\sqrt{1-\rho^2}}{\rho} > 0 \quad \forall \mathbf{x} \in \partial\mathcal{D}.$$

Thus we see that both Hypothesis **H2** and the inequality (2.15) holds true. .

To verify that the U -monotonicity condition is satisfied, observe that

$$\begin{aligned} \lambda_{\mathbf{f}}(\varphi, \mathbf{x}) + \frac{1}{2} \langle \nabla U(\mathbf{x}), \mathbf{f}(\varphi, \mathbf{x}) \rangle = \\ -\frac{1}{u(\mathbf{x})} \left[\frac{a}{\|\mathbf{x} + a\mathbf{k}\|^3} - \langle \mathbf{E}(\varphi), \mathbf{k} \rangle \right] = -\frac{1}{u(\mathbf{x})} \left[\frac{a}{(1 - 2au(\mathbf{x}) + a^2)^{3/2}} - \langle \mathbf{E}(\varphi), \mathbf{k} \rangle \right] \geq \\ \frac{a}{(1 + a)^3} - \langle \mathbf{E}(\varphi), \mathbf{k} \rangle > 0. \end{aligned}$$

Since it has been already shown that (2.10) is fulfilled, we complete the proof by applying Theorem (5) \square

Now let us proceed to the system perturbed by magnetic field and damping. Define

$$E := \max_{\varphi \in \mathbb{T}^k} \|\mathbf{E}(\varphi)\|, \quad \beta := \max_{\varphi \in \mathbb{T}^k} \|\mathbf{B}(\varphi)\| / \rho, \quad \varkappa := \kappa / \rho.$$

In order to make the results concerning the perturbed system more demonstrative, consider the case where $E_{\mathbf{k}}(\varphi) = 0$, $\mathbf{B}(\varphi) \perp \mathbf{E}(\varphi)$, i.e. $\mathbf{B}(\varphi) = B(\varphi)\mathbf{k}$ where $B(\cdot) : \mathbb{T}^k \mapsto \mathbb{R}$ is a smooth function (the typical case of crossed electric and magnetic field), and $\varkappa \leq \beta$. It is not hard to show that when applying the results of Section 2 one can set

$$\begin{aligned} M_f(\mathbf{x}) &:= \|\mathbf{x} + a\mathbf{k}\|^{-2} + E, \\ C_f &:= \frac{1}{2} \left[\max \left\{ (1 + 2as + a^2)^{-1} : \rho \leq s \leq 1 \right\} + E \right] = \\ &\quad \frac{1}{2} \left[E + (1 + 2a\rho + a^2)^{-1} \right] \leq \frac{1}{2} \left[E + (1 + a^2)^{-1} \right], \\ M_U(\mathbf{x}) &:= \frac{2\sqrt{1 - \langle \mathbf{k}, \mathbf{x} \rangle^2}}{\langle \mathbf{k}, \mathbf{x} \rangle}, \quad C_U := \max_{\rho \leq s \leq 1} \frac{2\sqrt{1 - s^2}}{s} = \frac{2\sqrt{1 - \rho^2}}{\rho}, \\ M_P(\varphi, \mathbf{x}) &:= \rho \sqrt{\beta^2 \langle \mathbf{k}, \mathbf{x} \rangle^2 + \varkappa^2}, \quad M_{PU}(\varphi, \mathbf{x}) := \frac{2\rho \sqrt{(1 - \langle \mathbf{k}, \mathbf{x} \rangle^2) (\beta^2 \langle \mathbf{k}, \mathbf{x} \rangle^2 + \varkappa^2)}}{\langle \mathbf{k}, \mathbf{x} \rangle}, \\ p &:= \max_{0 \leq s \leq 1} \sqrt{(1 - s^2) (\beta^2 s^2 + \varkappa^2)} = \frac{(\beta^2 + \varkappa^2)}{2\beta}, \quad l := \frac{(1 + a)^2 \varkappa \rho}{(1 + a)^2 E + 1}. \end{aligned}$$

(Here we use the equalities $\|\mathbf{e} \times \mathbf{k} - \langle \mathbf{e} \times \mathbf{k}, \mathbf{x} \rangle \mathbf{x}\|^2 = \|\mathbf{e} \times \mathbf{k}\|^2 - \langle \mathbf{e} \times \mathbf{k}, \mathbf{x} \rangle^2 = \langle \mathbf{k}, \mathbf{x} \rangle^2$.)

Let us evaluate $L_P(\varphi, \mathbf{x})$. Observe that a geodesic γ on \mathbb{S}^2 passing through a point \mathbf{x} at direction of a unite vector $\mathbf{e} \in T_{\mathbf{x}}\mathbb{S}^2$ coincides with an orbit of one-parameter subgroup $\{e^{\Omega t}\}_{t \in \mathbb{R}}$ of the group $\text{SO}(3)$. Such a geodesic is the rotation with angular velocity $\mathbf{w} := \mathbf{x} \times \mathbf{e}$, and Ω is a skew-symmetric operator such that $\Omega \mathbf{x} \equiv \mathbf{w} \times \mathbf{x}$. In addition, for any $\mathbf{e}_1 \in T_{\mathbf{x}_0}\mathbb{S}^2$ the mapping $t \mapsto e^{\Omega t} \mathbf{e}_1$ is the parallel translation along γ . Hence, for any $\mathbf{e}, \mathbf{e}_1 \in T_{\mathbf{x}}\mathbb{S}^2$, we have

$$\begin{aligned} |\langle \nabla P(\varphi, \mathbf{x})(\mathbf{e}, \mathbf{e}_1), \mathbf{e} \rangle| &= \left| \frac{d}{dt} \Big|_{t=0} \langle P(\varphi, e^{\Omega t} \mathbf{x}) e^{\Omega t} \mathbf{e}_1, e^{\Omega t} \mathbf{e} \rangle \right| = \\ &\quad \left| \frac{d}{dt} \Big|_{t=0} [\langle (e^{t\Omega} \mathbf{e}_1) \times \mathbf{B}(\varphi), e^{t\Omega} \mathbf{e} \rangle - \varkappa \langle e^{t\Omega} \mathbf{e}_1, e^{t\Omega} \mathbf{e} \rangle] \right| = \\ &\quad \left| \frac{d}{dt} \Big|_{t=0} \langle \mathbf{e}_1 \times (e^{-t\Omega} \mathbf{B}(\varphi)), \mathbf{e} \rangle \right| = |\langle \mathbf{e}_1 \times (\mathbf{B}(\varphi) \times \mathbf{w}), \mathbf{e} \rangle| = \\ &\quad \|\mathbf{B}(\varphi)\| |\langle \mathbf{k}, \mathbf{e} \rangle| |\langle \mathbf{x} \times \mathbf{e}, \mathbf{e}_1 \rangle|. \end{aligned}$$

Since the maximum is attained when $\mathbf{e}_1 = \mathbf{x} \times \mathbf{e}$, $\mathbf{e} = \mathbf{k} - \langle \mathbf{k}, \mathbf{x} \rangle \mathbf{x}$, we find

$$L_P(\varphi, \mathbf{x}) = \|\mathbf{B}(\varphi)\| \sqrt{1 - \langle \mathbf{k}, \mathbf{x} \rangle^2}$$

Thus conditions (2.8) and (2.16), respectively, take the form

$$\frac{a}{(1 + 2\rho a + a^2)^{3/2}} > \frac{E\rho}{\sqrt{1 - \rho^2}} + \frac{(\beta^2\rho^2 + \varkappa^2)\rho}{2},$$

$$\begin{aligned} & \frac{a}{(1 + 2as + a^2)^{3/2}} - z_*\rho\sqrt{(1 - s^2)} \left(2\sqrt{(\beta^2s^2 + \varkappa^2)} + \beta \right) - \\ & \frac{s\rho^2(\beta^2s^2 + \varkappa^2)}{4} > 0 \quad \forall s \in [\rho, 1]. \end{aligned}$$

Obviously, these inequalities are fulfilled once ρ is small enough to satisfy the condition

$$\frac{a}{(1 + a)^3} > \max \left\{ \rho z_* \left[\frac{2\beta^2 + \varkappa^2}{\beta} + \frac{\rho(\beta^2 + \varkappa^2)}{4} \right], \frac{\rho E}{\sqrt{1 - \rho^2}} + \frac{\rho(\beta^2\rho^2 + \varkappa^2)}{2} \right\}.$$

Observe that on account of Remark 3 we have

$$\rho z_* \leq \rho z_+ + \sqrt{\frac{\rho z_+}{q}} (1 - \rho^2)^{1/4} (1 + lz_+) \sqrt{\left(E + \frac{1}{1 + a^2} \right)}.$$

The above inequalities allow us to establish an upper bound for magnitude of perturbation which does not destroy the quasiperiodic solution obtained in Theorem 7.

It should be noted that in [2] the authors study trajectories of autonomous system governing the motion of classical particles accelerated by a potential and a magnetic field on a non-complete Riemannian manifold.

FINAL REMARKS

Since Lyapunov proposed his direct method, the analysis of nonlinear systems by means of auxiliary functions whose level sets are transversal to vector fields of systems' right hand sides was successfully carried out by many authors (see, e.g., [16, 17, 20, 21]). The success in constructing such functions for concrete systems depends on the art of researcher. In the case where the system (1.1) is a Lagrangian one, i.e. $f(\varphi, x) = -\nabla \Pi(\varphi, x)$, it is naturally to seek the auxiliary function $U(\cdot)$ using the averaged function $\int_{\mathbb{T}^k} \Pi(\varphi, x) d\varphi$, as was proposed in [24]. We will devote a separate paper to further applications of the results obtained.

REFERENCES

- [1] L. Amerio. Soluzioni quasi-periodiche, o limitate, di sistemi differenziali non lineari quasi-periodici, o limitati. (Italian) Ann. Mat. Pura Appl. (4) 39 (1955) 97–119.
- [2] R. Bartolo, A.V. Germinario, Trajectories of a charge in a magnetic field on Riemannian manifolds with boundary, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 17 (2010) no. 3 363–376.
- [3] M.S. Berger, Y.Y. Chen, Forced quasiperiodic and almost periodic solution for nonlinear systems, Nonlinear Anal. TMA 21 (1993) no. 12 949–965.
- [4] M. S. Berger, Luping Zhang, A new method for large quasiperiodic nonlinear oscillations with fixed frequencies for nondissipative second order conservative systems of second type, Commun. Appl. Nonlinear Anal. 3 (1996) no. 1 25–49.
- [5] J. Blot, Calculus of variations in mean and convex Lagrangians, J. Math. Anal. Appl. 134 (1988) no. 2 312–321.
- [6] J. Blot, Calculus of variations in mean and convex Lagrangians, II, Bull. Aust. Math. Soc. 40 (1989) no. 3 457–463.
- [7] J. Blot, Calculus of variations in mean and convex Lagrangians III, Israel J. Math. 67 (1989) no. 3 337–344.
- [8] J. Blot, P. Cieutat, J. Mawhin, Almost-periodic oscillations of monotone second-order systems, Adv. Differential Equations 2 (1997) no. 5 693–714.

- [9] J. Blot, D. Pennequin, Spaces of quasi-periodic functions and oscillations in differential equations, *Acta. Appl. Math* 65 (2001) no. 1–3 83–113.
- [10] C. Carminati, Forced systems with almost periodic and quasiperiodic forcing term, *Nonlinear Anal. TMA* 32 (1998) no. 6 727–739.
- [11] D. Cheban, C. Mammanna, Invariant manifolds, almost periodic and almost automorphic solutions of second-order monotone equations, *Int. J. Evol. Equ.* 1 (2005) no. 4 319–343.
- [12] V.M. Cheresiz, Stable and conditionally stable almost-periodic solutions of V-monotone systems, *Sib. Math. J.* 15 (1974) no. 1, 116–125.
- [13] P. Cieutat, Almost periodic solutions of second-order systems with monotone fields on a compact subset, *Nonlin. Anal.* 53 (2003) no. 6 751–763.
- [14] D. Gromoll, W. Klingenberg, W. Meyer, *Riemannsche Geometrie im Grossen* (German), Lecture Notes in Mathematics, No. 55, Springer-Verlag, Berlin-New York, 1968.
- [15] P. Hartman, *Ordinary differential equations*, John Wiley & Sons, Inc., New York-London-Sydney, 1964.
- [16] M. A. Krasnosel'skii, P. P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*, Berlin- Heidelberg-New York-Tokio: Springer-Verlag, A Series of Comprehensive Studies in Mathematics, 263, 1984.
- [17] A.M. Krasnosel'skii, M.A. Krasnosel'skii, J. Mawhin, A. Pokrovskii, Generalized guiding functions in a problem on high frequency forced oscillations, *Nonlinear Anal.* 22 (1994) no. 11 1357–1371.
- [18] J. Kuang, Variational approach to quasi-periodic solution of nonautonomous second-order Hamiltonian systems, *Abstr. Appl. Anal.*, 2012 (2012), Article ID 271616, 14 pp.
- [19] J. Mawhin, Bounded and almost periodic solutions of nonlinear differential equations: variational vs nonvariational approach, in: *Calculus of Variations and Differential Equations* (Haifa, 1998), Chapman & Hall/CRC Research Notes in Mathematics, 410, Boca Raton, FL, 2000, pp. 167–184.
- [20] J. Mawhin, J.R. Ward Jr. Guiding-like functions for periodic or bounded solutions of ordinary differential equations, *Discrete Contin. Dyn. Syst.* 8 (2002) no. 1 39–54.
- [21] V. Obukhovskii, P. Zecca, N. Van Loi, S. Kornev, *Method of guiding functions in problems of nonlinear analysis*, Lecture Notes in Mathematics 2076, Springer, Heidelberg, 2013.
- [22] R. Ortega, The pendulum equation: from periodic to almost periodic forcings, *Differential Integral Equations* 22 (2009) no. 9–10 801–814.
- [23] L.A. Pars, *A Treatise on Analytical Dynamics*, John Wiley & Sons, Inc., New York, 1965.
- [24] I. Parasyuk, A. Rustamova, Variational approach for weak quasiperiodic solutions of quasiperiodically excited Lagrangian systems on Riemannian manifolds, *Electronic Journal of Differential Equations* 2012 (2012) no. 66 22 pp.
- [25] I.O. Parasyuk, Quasiperiodic extremals of nonautonomous Lagrangian systems on Riemannian manifolds. Translation of *Ukrain. Mat. Zh.* 66 (2014) no. 10 1387–1406, *Ukrainian Math. J.* 66 (2015) no. 10 1553–1574.
- [26] A.M. Samoilenko, On the exponential dichotomy on \mathbb{R}^n of linear differential equations in \mathbb{R}^n . Translation of *Ukrain. Mat. Zh.* 53 (2001) no. 3 356–371, *Ukrainian Math. J.* 53 (2001) no. 3 407–426.
- [27] Yu.V. Trubnikov, A.I. Perov, *Differential equations with monotone nonlinearities* (Russian), Nauka i Tekhnika, Minsk, 1986.
- [28] S.F. Zakharin, I.O. Parasyuk, Generalized and classical almost periodic solutions of Lagrangian systems, *Funkcial. Ekvac.*, 42 (1999), 325–338.
- [29] S. F. Zakharin, I. O. Parasyuk, On the smoothness of generalized quasiperiodic solutions of Lagrangian systems on Riemannian manifolds of nonpositive curvature (Ukrainian), *Nelineini Kolyv.* 2 (1999) no. 2 180–193.

FACULTY OF MECHANICS AND MATHEMATICS, TARAS SHEVCHENKO NATIONAL UNIVERSITY OF KYIV,
 64/13, VOLODYMYRSKA STREET, CITY OF KYIV, UKRAINE, 01601
E-mail address: pio@univ.kiev.ua